Quantum Field Theory in Curved Spacetime

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Contents

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QFT in Curved Spacetime

In this course, we will be interested in understanding quantum field theory in the presence of a non-trivial gravitational field. We will consider the gravitational field as a classical fixed background on which our quantum fields live. Towards the end of the course, we will also study quantum fluctuations of the background itself, but in order for the perturbative quantum field theory to be consistent, we will always consider energy scales small compared to the Planck scale. Later it will become clear that, in order to understand quantum gravity at the Planck scale, we will need a UV (ultraviolet) completion of our effective quantum field theory treatment, such as string theory.

The effective QFT approach to quantum gravity presented here is, however, enough to understand the quantum evolution of the universe from a few Planck times after the Big Bang and until today. We will develop the machinery for understanding how the universe was created from quantum fluctuations during inflation (or the alternative Pre-Big Bang scenarios). It will also allow us to understand how Hawking radiation is created in the weak curvature regime outside a black hole and arrive at the black hole information paradox. These discoveries are all a consequence of the fact that the concept of particles and the notion of vacuum are non-unique and relative concepts in QFT in curved space, and may change with time, leading f.ex. to particle production in the early universe as it expands with time and in the vicinity of a black hole.

These are some of the puzzling and counter-intuitive concepts we will cover in these lectures.

Canonical Quantization

Before we proceed to understand quantum field theory on a curved background, let us, for a start, remind ourselves about the simplest possible quantum field theory, a free real scalar field on a flat background. It is often, correctly, said that a field can be viewed as a collection of infinitely many harmonic oscillators. The quantisation of a field therefore has some resemblance to quantising the harmonic oscillator. So let us remind ourselves how it goes.

1.1 Quantization of the Harmonic Oscillator

In the case of a classical harmonic oscillator, we have the equation of motion (EoM):

$$\ddot{q} + \omega^2 q = 0 \ , \tag{1.1}$$

where the mass, m, is absorbed into the eigen-frequency $\omega = \sqrt{k/m}$, with k being the spring constant in Hooks law. The general solution can be written as:

$$q(t) = a^- e^{-i\omega t} + a^+ e^{i\omega t} \tag{1.2}$$

It is a second-order differential equation, so we have two integration constants, a^- and a^+, while ω is the frequency of the oscillator. For a real classical coordinate, the integration constants obey $a^+ = (a^-)^*$. The lowest energy state is called the ground state, and in the case of the classical harmonic oscillator, it is given by $q(t) = 0$. While this is intuitively clear, we can see it more formally by considering the Hamiltonian

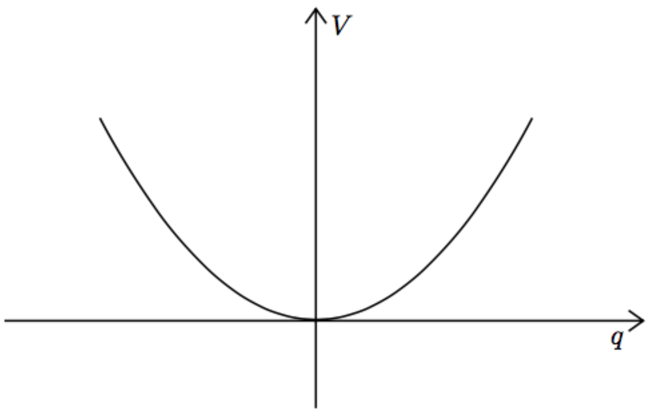


Figure 1: Illustration of the potential for the harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \tag{1.3}$$

where $p = \dot{q}$ is the momentum. The Hamiltonian gives the total energy of the system and is the sum of a kinetic energy term (the momentum term), and a quadratic potential term proportional to the frequency. Clearly, the Hamiltonian vanishes in the ground state with $q(t) = 0$ and thus also $p = 0$.

Often we will consider the action, expressed in terms of the Lagrangian, as a more fundamental object defining our physical system, as it can be written without reference to a preferred time, and therefore takes a more manifestly covariant form. However, for the purpose of canonical quantisation, it is convenient to choose a slicing of the space-time and define a preferred time variable. Technically, this is most adequately done in the important ADM formalism, which we will discuss later, and which was developed by Arnowitt, Deser, and Misner in a kindergarten on the Danish Island of Bornholm. But for now, when considering the Harmonic oscillator, there is no ambiguity in the choice of the time variable, and we can straightforwardly derive the Hamiltonian from the Lagrangian.

The action, S, written in terms of the Langrangian, L, is for the harmonic oscillator

$$S = \int L\left(q, \dot{q}\right) dt = \int \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2\right] dt \ . \tag{1.4}$$

Using the variational principle, one arrives as usual, at the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{1.5}$$

leading to Eq. (1.1). Equivalently, we can also derive the EOM from the Hamiltonian. Defining the canonical momentum

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} \ , \tag{1.6}$$

the Hamiltonian is defined as a Legendre transformation of the Lagrangian

$$H = \left(p\dot{q} - L\right)_{\dot{q}\to p} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \tag{1.7}$$

The Hamiltonian equations of motion, equivalent to the Euler-Lagrange equation, are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{1.8}$$

which, for the Harmonic oscillator, becomes

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q \ . \tag{1.9}$$

In order to quantise the Harmonic oscillator, we follow the usual quantisation procedure. We leave the equation of motion unchanged, but the coordinates become operators $q\left(t\right), p\left(t\right) \to \hat{q}\left(t\right), \hat{p}\left(t\right)$, and then we postulate Heisenberg's commutation relation

$$\left[\hat{q}\left(t\right), \hat{p}\left(t\right)\right] = i\hbar \ . \tag{1.10}$$

Finally, we observe that the Hamiltonian now also has to be promoted into an operator

$$H \to \hat{H} = H\left(\hat{q}, \hat{p}\right) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 \ . \tag{1.11}$$

Quantum ground state

When looking for the ground state of the quantised system, we notice that due to Heisenberg's commutation relation, $\hat{q}(t) = 0$ is forbidden, and so it is not a ground state. The ground state has been lifted. We can see this more precisely by introducing the annihilation and creation operators

$$\hat{a}^{\pm}(t) = \sqrt{\frac{\omega}{2\hbar}} \left[\hat{q}(t) \mp \frac{i}{\omega} \hat{p}(t) \right] \tag{1.12}$$

With the annihilation and creation operators, we can write the Hamiltonian as

$$\hat{H} = \left(\hat{a}^{+} \hat{a}^{-} + \frac{1}{2} \right) \omega \hbar \ . \tag{1.13}$$

An energy eigenstate is of course an eigenstate of the Hamiltonian, and therefore an eigenstate of the number operator $N \equiv \hat{a}^{+}\hat{a}^{-}$. We usually denote these eigenstates as $|n\rangle$, with non-negative eigenvalues n. It is easy to see that n is non-negative, since we can write $n = \langle n|N|n\rangle = ||\hat{a}^{-}|n\rangle||^2$.

From these observations, it is clear that the ground state (the vacuum state) must be defined as $\hat{a}^{-}|0\rangle = 0$, and we obtain a non-zero vacuum energy

$$\langle 0|\hat{H}|0\rangle = \frac{1}{2}\omega\hbar \ . \tag{1.14}$$

Thus the quantum harmonic oscillator is never at rest, but always fluctuates around the minimum of the potential, even in its ground state.

As mentioned earlier, we can think of a quantum field as an infinite collection of quantum harmonic oscillators, so the above result already hints at a deep and fundamental problem of understanding the smallness of the vacuum energy of our universe. For reasons that will become clear later, this is also named the cosmological constant problem.

1.2 Relativistic Fields

So far our considerations have been contained within the domain of non-relativistic quantum mechanics. In the absence of a potential, the Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x,t) \ . \tag{1.15}$$

With the usual position-representation identifications

$$\hat{p}_j = -i\hbar \frac{\partial}{\partial x_j}, \qquad \hat{E} = i\hbar \frac{\partial}{\partial t} \tag{1.16}$$

the Schrödinger equation for the eigenvalues takes the form

$$E = \frac{p^2}{2m} \ , \tag{1.17}$$

while special relativity requires

$$\frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2 \ . \tag{1.18}$$

Clearly, the Schrödinger equation does not satisfy the relativistic energy-momentum relation. If we were to use (1.16) in the relativistic energy-momentum relation in (1.18), we would get

$$\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\nabla^2\right)\Psi = m^2c^2\Psi \ . \tag{1.19}$$

This is known as the Klein-Gordon equation, and the eigenvalue equation now agrees with the relativistic energy-momentum relation.

If the Klein-Gordon equation is the correct relativistic version of the Schrödinger equation, then we should be able to recover the Schrödinger equation in the non-relativistic limit. Writing

$$E = \pm\sqrt{p^2c^2 + m^2c^4} \ , \tag{1.20}$$

choosing only positive energy, and Taylor expanding for small $\boldsymbol{p}^2$ we get

$$E = mc^2 + \frac{\boldsymbol{p}^2}{2m} + \ldots \tag{1.21}$$

The extra mass term mc^2 becomes irrelevant in non-relativistic quantum mechanics in the absence of gravity, since the Hamiltonian becomes an operator, and it is just an arbitrary additive constant

$$\hat{H} = \frac{\hat{p}^2}{2m} + mc^2 = \frac{\hat{p}^2}{2m} + const \tag{1.22}$$

which will cancel out in all commutation relations, and therefore will not influence quantum dynamics. The negative-frequency solution appears to spoil the interpretation of the $\Psi\,(x,t)$ as a probability amplitude density. In quantum field theory, positive- and negative-frequency solutions are instead associated with annihilation and creation operators. For a complex scalar field, the latter are naturally associated with antiparticles, while a real scalar particle is its own antiparticle.

The above relativistic generalisation of the Schrödinger equation is not unique. There are several ways to generalise the Schrödinger equation to obtain a relativistic covariant equation, depending on the spin of the quantum field.
Wigner classified relativistic one-particle states in terms of unitary irreducible representations of the Poincaré group in 1939. Some well-known physical examples are given in the table below

Spin	Wave equation	examples
0	Klein-Gordon	Higgs, Inflaton, π^0, $\pi^\pm$
$\frac{1}{2}$	Dirac	e^-, quarks, p^+, n
1	Maxwell/Yang–Mills	Photons, gluons

Can we have higher spins? No elementary particles with spin greater than one have been observed. Interacting higher-spin fields can be described within effective field theories, although consistent interactions become increasingly constrained for higher spins. In particular, general relativity describes a massless spin-2 graviton and is non-renormalizable by power counting when treated perturbatively as a fundamental quantum field theory, while remaining predictive as a low-energy effective field theory. The Einstein equation is non-linear and does not lead to an additive superposition of the wave functions. When quantising the graviton, we will therefore linearise the Einstein equation around some background, and the equations for the different spin modes of the linearised graviton will then reduce to something similar to the Klein-Gordon equation. Therefore, the main focus of these lectures is on the Klein-Gordon equation in curved space, although we will also discuss fermions and gauge fields. The generalisation to free relativistic fields of arbitrary spin is possible through the Bargmann-Wigner equations.

From now on, we use natural units $\hbar = c = 1$.

Now, let us return to consider the Klein-Gordon equation:

$$\ddot{\phi} - \nabla^2\phi + m^2\phi \equiv -\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{1.23}$$

where, with the $(-+++)$ convention, $\partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$. Equivalently, the covariant form of the equation is $\left(\partial_\mu\partial^\mu - m^2\right)\phi = 0$.

Using spatial Fourier decomposition

$$\phi\left(\boldsymbol{x}, t\right) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{\boldsymbol{k}}\left(t\right) \tag{1.24}$$

we obtain

$$\ddot{\phi}_{\boldsymbol{k}} + \left(\boldsymbol{k}^2 + m^2\right)\phi_{\boldsymbol{k}} = \ddot{\phi}_{\boldsymbol{k}} + \omega_{\boldsymbol{k}}^2\phi_{\boldsymbol{k}} = 0 \ , \tag{1.25}$$

where $\omega_{\boldsymbol{k}} \equiv \sqrt{\boldsymbol{k}^2 + m^2}$. Notice that we have a harmonic oscillator for each Fourier mode. Modes of a real scalar field satisfy the relation $(\phi_{\boldsymbol{k}})^* = \phi_{-\boldsymbol{k}}$.
The equation of motion can be found from the action:

$$\begin{aligned} S[\phi] &= -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + m^2\phi^2\right] \\ &\equiv \frac{1}{2}\int d^3\boldsymbol{x}dt \left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right] \end{aligned} \tag{1.26}$$

With the Minkowski metric $\eta^{\mu\nu} = diag(-1, +1, +1, +1)$.

$$S = \frac{1}{2}\int dtd^3\boldsymbol{k}\left[\dot{\phi}_{\boldsymbol{k}}\dot{\phi}_{-\boldsymbol{k}} - \omega_{\boldsymbol{k}}^2\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}\right] \tag{1.27}$$

Quantization of Real Scalar Fields

In order to quantise the scalar field, we follow the quantisation procedure similar to how we quantised the harmonic oscillator. In order to go to the Hamiltonian formulation, we first read off the Lagrangian from the action above

$$L(\phi) = \int d^3\boldsymbol{x}\mathcal{L}, \quad \mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 \tag{1.28}$$

and introduce the canonical conjugate field (the field equivalent to the canonical conjugate momentum)

$$\pi(\boldsymbol{x}, t) \equiv \frac{\delta L[\phi]}{\delta\dot{\phi}(\boldsymbol{x}, t)} = \dot{\phi}(\boldsymbol{x}, t) \ , \tag{1.29}$$

which enables us to find the classical Hamiltonian by a Legendre transformation as before

$$H = \int d^3\boldsymbol{x}\pi(\boldsymbol{x}, t)\dot{\phi}(\boldsymbol{x}, t) - L\big|_{\dot{\phi}\to\pi} = \frac{1}{2}\int d^3\boldsymbol{x}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{1.30}$$

To quantise, we now promote the field and the conjugate momenta to operators ($\phi, \pi \to \hat{\phi}, \hat{\pi}$), and impose the commutator relations:

$$\left[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \quad \left[\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{y}, t)\right] = [\hat{\pi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)] = 0 \tag{1.31}$$

The Fourier modes also become operators with the commutator relations

$$\left[\hat{\phi}_{\boldsymbol{k}}(t), \hat{\pi}_{\boldsymbol{k}'}(t)\right] = i\delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}') \tag{1.32}$$

Notice the sign in the delta function. This is because the conjugate to $\hat{\phi}_{\boldsymbol{k}}$ is not $\hat{\pi}_{\boldsymbol{k}}$, but $\hat{\pi}_{-\boldsymbol{k}} = \hat{\pi}^{\dagger}_{\boldsymbol{k}}$.

One can also immediately see that the Hamiltonian equations of motion in operator form, become the Heisenberg equations of motion

$$
\begin{aligned}
i\dot{\hat{\phi}}\left(\boldsymbol{x}, t\right) &= \left[\hat{\phi}\left(\boldsymbol{x}, t\right), \hat{H}\right] \\
i\dot{\hat{\pi}}\left(\boldsymbol{x}, t\right) &= \left[\hat{\pi}\left(\boldsymbol{x}, t\right), \hat{H}\right] \ .
\end{aligned}
\tag{1.33}
$$

Hermiticity

Since our field $\phi(x,t)$ is a real scalar field, we have immediately

$$
\hat{\phi}^{\dagger}\left(\boldsymbol{x}, t\right) = \hat{\phi}\left(\boldsymbol{x}, t\right), \quad \hat{\pi}^{\dagger}\left(\boldsymbol{x}, t\right) = \hat{\pi}\left(\boldsymbol{x}, t\right) \ , \tag{1.34}
$$

but as noted already above, the Fourier modes are not Hermitian

$$
\hat{\phi}^{\dagger}_{\boldsymbol{k}}\left(t\right) = \hat{\phi}_{-\boldsymbol{k}}\left(t\right), \quad \hat{\pi}^{\dagger}_{\boldsymbol{k}}\left(t\right) = \hat{\pi}_{-\boldsymbol{k}}\left(t\right) \tag{1.35}
$$

as can be seen by inspecting the Fourier integrals.

$$
\begin{aligned}
\hat{\phi}^{\dagger}_{\boldsymbol{k}}\left(t\right) &= \int \frac{d^3\boldsymbol{x}}{\left(2\pi\right)^{3/2}} \left(e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}\hat{\phi}\left(\boldsymbol{x}, t\right)\right)^{\dagger} = \int \frac{d^3\boldsymbol{x}}{\left(2\pi\right)^{3/2}} e^{i\boldsymbol{x}\cdot\boldsymbol{k}}\hat{\phi}^{\dagger}\left(\boldsymbol{x}, t\right) \\
&= \int \frac{d^3\boldsymbol{x}}{\left(2\pi\right)^{3/2}} e^{i\boldsymbol{x}\cdot\boldsymbol{k}}\hat{\phi}\left(\boldsymbol{x}, t\right) = \hat{\phi}_{-\boldsymbol{k}}\left(t\right)
\end{aligned}
\tag{1.36}
$$

For each pair of Fourier modes $\boldsymbol{k}$ and $-\boldsymbol{k}$, one may form Hermitian combinations corresponding to the real and imaginary parts of $\hat{\phi}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$. These combinations can be normalised to form canonical position and momentum operators. Since $\boldsymbol{k}$ and $-\boldsymbol{k}$ are not independent for a real field, one must restrict to one representative of each pair in such a description in order not to double count the degrees of freedom.

However, in the end it is more convenient to work with the non-Hermitian Fourier modes $\hat{\phi}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$, which are straightforward Fourier transforms of the real space fields.

Quantum Field Oscillators

Let us consider again the field's Fourier mode

$$
\hat{\phi}_{\boldsymbol{k}}\left(t\right) = \int \frac{d^3\boldsymbol{x}}{\left(2\pi\right)^{3/2}} e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}\hat{\phi}\left(\boldsymbol{x}, t\right) \tag{1.37}
$$

Just like for the Harmonic oscillator, it is convenient to introduce creation and annihilation operators in order to understand the quantum states of the system

$$\hat{a}^-_{\boldsymbol{k}}(t) \equiv \sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}\left(\hat{\phi}_{\boldsymbol{k}} + \frac{i\hat{\pi}_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\right), \quad \hat{a}^+_{\boldsymbol{k}}(t) \equiv \sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}\left(\hat{\phi}_{-\boldsymbol{k}} - \frac{i\hat{\pi}_{-\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\right) \tag{1.38}$$

The equation of motion of the fields can then be equivalently expressed as an equation of motion for the creation and annihilation operators

$$\frac{d}{dt}\hat{a}^\pm_{\boldsymbol{k}}(t) = \pm i\omega_{\boldsymbol{k}}\hat{a}^\pm_{\boldsymbol{k}}(t) \tag{1.39}$$

which has a straightforward solution

$$\hat{a}^\pm_{\boldsymbol{k}}(t) = \hat{a}^\pm_{\boldsymbol{k}}(t=0)\, e^{\pm i\omega_{\boldsymbol{k}} t} \tag{1.40}$$

There are two independent solutions for each frequency mode, and these solutions form a plane wave basis for the general solutions of the equation of motion, defined by what we can think of as the two constants of integration $\hat{a}^\pm_{\boldsymbol{k}}(t=0)$.

Inserting into the canonical commutation relations for the fields, one can verify that the time-independent operators $\hat{a}^\pm_{\boldsymbol{k}} = \hat{a}^\pm_{\boldsymbol{k}}(t=0)$ must satisfy

$$\left[\hat{a}^-_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}'}\right] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'), \quad \left[\hat{a}^-_{\boldsymbol{k}}, \hat{a}^-_{\boldsymbol{k}'}\right] = \left[\hat{a}^+_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}'}\right] = 0 \; . \tag{1.41}$$

Clearly, the ground state, $|0\rangle$, will be the state where each of the harmonic oscillators is in its ground state and is defined by

$$\hat{a}^-_{\boldsymbol{k}}|0\rangle = 0 \quad \forall \boldsymbol{k} \tag{1.42}$$

We can now build the higher excited states by acting with the creation operator as usual. A state with particle number n_s in each mode with momentum $\boldsymbol{k}_s$ is given by

$$|n_1, n_2, \ldots\rangle = \left[\prod_s \frac{\left(\hat{a}^+_{\boldsymbol{k}_s}\right)^{n_s}}{\sqrt{n_s!}}\right]|0\rangle \; , \tag{1.43}$$

and these states make up the Fock Space.

Writing the Fourier modes in terms of creation and annihilation operators

$$\begin{aligned}
\hat{\phi}_{\boldsymbol{k}} &= \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(\hat{a}^-_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t} + \hat{a}^+_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t}\right) \\
\hat{\pi}_{\boldsymbol{k}} &= i\sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}\left(\hat{a}^+_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t} - \hat{a}^-_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t}\right) \; ,
\end{aligned} \tag{1.44}$$

we can then express the Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3\boldsymbol{k}\left[\hat{\pi}_{\boldsymbol{k}}\hat{\pi}_{-\boldsymbol{k}} + \omega_{\boldsymbol{k}}^2\hat{\phi}_{\boldsymbol{k}}\hat{\phi}_{-\boldsymbol{k}}\right] \tag{1.45}$$

in terms of the creation and annihilation operators

$$\hat{H} = \int d^3\boldsymbol{k}\frac{\omega_{\boldsymbol{k}}}{2}\left[\hat{a}_{\boldsymbol{k}}^-\hat{a}_{\boldsymbol{k}}^+ + \hat{a}_{\boldsymbol{k}}^+\hat{a}_{\boldsymbol{k}}^-\right] \tag{1.46}$$

and after deploying the commutator relation, it takes the form

$$\hat{H} = \int d^3\boldsymbol{k}\frac{\omega_{\boldsymbol{k}}}{2}\left[2\hat{a}_{\boldsymbol{k}}^+\hat{a}_{\boldsymbol{k}}^- + \delta^{(3)}(0)\right] \ , \tag{1.47}$$

where we got a delta function from the commutator, and this term is the vacuum energy.

$$E = \int d^3\boldsymbol{k}\omega_{\boldsymbol{k}}N_{\boldsymbol{k}} + vacuum\ energy \tag{1.48}$$

Where $E = \left\langle \hat{H} \right\rangle$ and $N_{\boldsymbol{k}}$ comes from the number operator:

$$N_{\boldsymbol{k}} = \left\langle \hat{N}_{\boldsymbol{k}} \right\rangle = \left\langle \hat{a}_{\boldsymbol{k}}^+\hat{a}_{\boldsymbol{k}}^- \right\rangle \tag{1.49}$$

With the Fourier convention used here, $\delta^{(3)}(0) = V/(2\pi)^3$ in a finite-volume regularization. Thus the formal vacuum energy density is

$$\rho_{vac} = \frac{E_{vac}}{V} = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\omega_{\boldsymbol{k}}, \tag{1.50}$$

which is ultraviolet divergent.

So far, we have quantised $\phi(\boldsymbol{x}, t)$ in the Heisenberg picture. Quantum observables $(\hat{\phi}, \hat{H})$ are represented as operators in Fock space, and the quantum states of the field ϕ are interpreted in terms of particles (i.e. $|n_1, n_2, \ldots\rangle$). In units of $\hbar = c = 1$, the particle interpretation is consistent with

$$\boldsymbol{p} = \boldsymbol{k}, \quad E = \omega_{\boldsymbol{k}} \Rightarrow E = \sqrt{p^2 + m^2}\,, \tag{1.51}$$

which is the relativistic expression of the energy of a particle.

Mode Expansions

The creation and annihilation operators also provide us with a shortcut quantisation. We can directly write the field in the complete plane wave basis, with two "integration constants"

$$\hat{\phi}_{\boldsymbol{k}}(t) = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\hat{a}^-_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t} + \hat{a}^+_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t} \right) \tag{1.52}$$

and then go back to the real-space representation of the field

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\hat{a}^-_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^+_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} \right) \ .$$

Then taking $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term gives

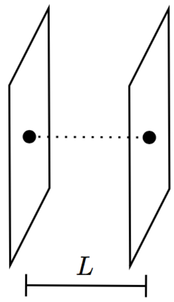
$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(\hat{a}^-_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^+_{\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}} \right) \tag{1.53}$$

as a general solution to the Klein-Gordon equation where one can think of $a^\pm$ as constants of integration.

All we need to do to quantise the field in terms of the mode expansion is to introduce the commutator relations:

$$\left[\hat{a}^-_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}'}\right] = \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right), \quad \left[\hat{a}^\pm_{\boldsymbol{k}}, \hat{a}^\pm_{\boldsymbol{k}'}\right] = 0 \tag{1.54}$$

Casimir Effect



If we assume that vacuum energy is like any other form of energy, then within the theory of General Relativity it will gravitate, and the infinite vacuum energy density will lead to an infinite gravitational field, which would imply that our universe, as we know it, could not exist. Clearly, the vacuum energy is not really infinite, or General Relativity is wrong, and vacuum energy does not gravitate like any other form of energy. To get insight into what is going on and what we should believe, let us consider the Casimir effect.

We have already observed that a quantum field can be seen as a collection of infinitely many harmonic oscillators. Since the vacuum energy of a single harmonic oscillator is different from zero, we get a divergent contribution to the vacuum energy in quantum field theory. More precisely, in infinite Minkowski space the zero-point energy density of a free scalar field is formally

$$\rho_0 \equiv \frac{E_0}{V} = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\omega_{\boldsymbol{k}} \tag{2.1}$$

which is ultraviolet divergent.

One could argue heuristically that there is an upper limit to the momentum integral: attempting to localise an energy much larger than the Planck scale within its corresponding Compton wavelength eventually gives a Schwarzschild radius larger than that wavelength, suggesting that ordinary local quantum field theory cannot be trusted beyond the Planck scale. Introducing a UV momentum cutoff $\Lambda \sim M_{Pl}$, one still obtains a fatal high vacuum energy density $\rho \sim M_{Pl}^4$.

When we are cutting off the momentum integral by introducing the UV cutoff Λ, we are secretly *regularising* the integral, and we obtain a finite but large value, which diverges as we remove the cutoff by taking $\Lambda \to \infty$. This is actually the first step in how we deal with fatal divergent quantities in quantum field theory in general through the procedure of regularisation and renormalisation. The second step, *renormalisation*, is a prescription for how to deal with the subsequent large arbitrary cutoff-dependent but finite contribution to the physical quantity (in the present case the cosmological constant).

To get some intuition for the process of regularization and renormalization, and an insight into the physical realm of the vacuum energy of quantum fields, we will consider the Casimir effect, an experimentally verified consequence of quantum field theory in the presence of boundaries. It provides a particularly clean example in which a divergent zero-point energy can be used to obtain a finite, regulator-independent energy difference.

Consider a massless scalar $\phi(t, \boldsymbol{x})$ in flat 1+1 dimensional space-time.[2] We require the field to vanish on the boundaries by imposing boundary conditions

$$\phi(t,x)\,|_{x=0} = \phi(t,x)\,|_{x=L} = 0\ . \tag{2.2}$$

The equation of motion for a massless scalar field in one dimension is

$$\partial_t^2\phi - \partial_x^2\phi = 0 \tag{2.3}$$

and a general solution satisfying the boundary conditions is then given by

$$\phi(t,x) = \sum_{n=1}^{\infty}\left(A_n e^{-i\omega_n t} + B_n e^{i\omega_n t}\right)\sin(\omega_n x) \tag{2.4}$$

with $\omega_n \equiv \frac{n\pi}{L}$. From this solution we can construct an appropriate basis for the mode expansion of the scalar field.

Remember that in flat space (without boundary conditions), we have the following mode expansion in the plane wave basis

$$\hat{\phi}(t,x) = \int \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega_k}}\left[\hat{a}_k^- e^{-i\omega_k t + ikx} + \hat{a}_k^+ e^{i\omega_k t - ikx}\right]\ . \tag{2.5}$$

However, the plane waves modes does not satisfy the boundary conditions. Instead of plane wave modes, the boundary conditions leads to a discrete subset of orthonormal functions, which forms the basis for the mode expansion. Normalizing the modes of the solution (2.4), we are led to introduce the orthonormal basis functions

$$\int_0^L g_m(x)\, g_n(x)\, dx = \delta_{mn}, \qquad g_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \tag{2.6}$$

[2]This is a simplified setup decribing the vacuum energy of the electromagnetic field between two conducting plates.

The continuous wavenumber, k, is replaced with a discrete one, $\frac{n\pi}{L}$. The orthonormal basis functions, $g_n(x)$, satisfies the boundary conditions and vanishes at $x = 0$ and $x = L$.

Expanding the field in the $\exp(-i\omega_n t)g_n(x)$ basis instead of the plane wave basis, we obtain in the presence of the boundary conditions

$$\hat{\phi}\,(t,x) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\frac{\sin\left(\omega_n x\right)}{\sqrt{2\omega_n}}\left[\hat{a}_n^- e^{-i\omega_n t} + \hat{a}_n^+ e^{i\omega_n t}\right] \tag{2.7}$$

with $\omega_n \equiv \frac{n\pi}{L}$. We observe that fewer modes are allowed between the plates than outside. Only those waves with a wavelength just right to form a standing wave between the plates are allowed. Thus, instead of having a continuous spectrum of virtual particles being created in vacuum, between the plates only a discrete subset is allowed. Intuitively, this makes the vacuum energy smaller between the plates than outside.

2.1 The Zero-Point Energy

In order to compute the vacuum energy between the plates, let us consider the Hamiltonian

$$\hat{H} = \frac{1}{2}\int_0^L dx\left[\left(\partial_t\hat{\phi}\right)^2 + \left(\partial_x\hat{\phi}\right)^2\right] \tag{2.8}$$

The vacuum energy is found by taking the expectation value in the ground state $\langle 0|\hat{H}|0\rangle$, and using

$$\langle 0|\hat{a}_m^-\hat{a}_n^+|0\rangle = \delta_{mn}, \quad \langle 0|\hat{a}_m^+\hat{a}_n^+|0\rangle = \langle 0|\hat{a}_m^-\hat{a}_n^-|0\rangle = \langle 0|\hat{a}_m^+\hat{a}_n^-|0\rangle = 0 \tag{2.9}$$

which yields from the first term in $\hat{H}$

$$\begin{aligned}\langle 0|\frac{1}{2}\int_0^L dx\left(\partial_t\hat{\phi}\right)^2|0\rangle &= \langle 0|\frac{1}{2}\int_0^L dx\left[\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\frac{\sin\left(\omega_n x\right)}{\sqrt{2\omega_n}}i\omega_n\left(-\hat{a}_n^- e^{-i\omega_n t} + \hat{a}_n^+ e^{i\omega_n t}\right)\right]^2|0\rangle \\ &= \frac{1}{L}\int_0^L dx\sum_{n=1}^{\infty}\frac{\left(\sin\left(\omega_n x\right)\right)^2}{2\omega_n}\omega_n^2 = \frac{1}{4}\sum_n\omega_n \end{aligned} \tag{2.10}$$

The second term in $\hat{H}$ yields the same contribution, which leaves us with the zero point energy

$$\langle 0|\hat{H}|0\rangle = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n \tag{2.11}$$

To compute something like an energy density, we can compute the zero point energy per unit length

$$\varepsilon_0 \equiv \frac{1}{L}\langle 0|\hat{H}|0\rangle = \frac{1}{2L}\sum_n \omega_n = \frac{\pi}{2L^2}\sum_{n=1}^{\infty} n \tag{2.12}$$

As we can see, this is again divergent. To deal with it, we turn to regularization and renormalization.

2.2 Regularization and Renormalization

The expression $\varepsilon_0(L)$ in (2.12) is divergent, so neither it nor its $L \to \infty$ limit can be manipulated directly. The meaningful comparison is obtained by regulating both the finite-L and free-space expressions in the same way and then subtracting them. In nongravitational flat-space QFT an overall additive constant in the energy is unobservable, so it is conventional to choose the free-space vacuum energy as the reference value. This should not be confused with a prediction that the physical cosmological constant must vanish.

We are therefore led to compare ε_0^{free} with $\varepsilon_0(L)$:

$$\Delta\varepsilon\left(L\right) = \varepsilon_0\left(L\right) - \varepsilon_0^{free} \tag{2.13}$$

Of course $\Delta\varepsilon(L) = \infty - \infty$ make little sense. In order to make sense of this difference, we first need to regulate $\varepsilon_0(L)$ and ε_0^{free} by a cutoff.

The procedure is therefore: first regulate both quantities using the same regulator; second form the difference so that the common divergent contribution cancels (equivalently, absorb the divergence into the appropriate counterterm); and finally remove the regulator. A physical result should be independent of the arbitrary details of the regulator.

Instead of introducing a hard cutoff, we will regularize $\varepsilon_0(L)$ by introducing a cutoff function, which exponentially truncates the UV modes

$$\varepsilon_0\left(L\right) = \frac{\pi}{2L^2}\sum_{n=1}^{\infty} n \quad \Rightarrow \quad \varepsilon_0\left(L;\alpha\right) = \frac{\pi}{2L^2}\sum_{n=1}^{\infty} n \exp\left(-\frac{n\alpha}{L}\right) \tag{2.14}$$

where α is the cutoff parameter. For $\alpha > 0$ the regularized series converges, and we can recover the original expression by taking $\alpha \to 0$. To see this, lets write

$$\begin{aligned}\varepsilon_0(L,\alpha) &= -\frac{\pi}{2L}\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty}\exp\left[-\frac{n\alpha}{L}\right] \\ &= -\frac{\pi}{2L}\frac{\partial}{\partial\alpha}\left(\frac{\exp\left[-\frac{\alpha}{L}\right]}{1-\exp\left[-\frac{\alpha}{L}\right]}\right) = \frac{\pi}{2L^2}\frac{\exp\left(-\frac{\alpha}{L}\right)}{\left[1-\exp\left[-\frac{\alpha}{L}\right]\right]^2} \\ &= \frac{\pi}{8L^2}\frac{1}{\sinh^2\left(\frac{\alpha}{2L}\right)}\ ,\end{aligned} \tag{2.15}$$

where we used

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{2.16}$$

so we can write

$$\frac{1}{\sinh^2\left(\frac{x}{2}\right)} = \frac{4}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2} = \frac{4}{e^x\left(1-e^{-x}\right)^2} = \frac{4e^{-x}}{\left(1-e^{-x}\right)^2} \tag{2.17}$$

Taking the limit $\alpha \to 0$, we can use the expansion of $\sinh(x)$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = x\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right) \tag{2.18}$$

which gives us the expansion

$$\frac{1}{\sinh^2(x)} = \frac{1}{x^2\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\right)^2} = \frac{1}{x^2}\left(1 - 2\frac{x^2}{3!} + \ldots\right)\ . \tag{2.19}$$

Thus, in the limit $\alpha \to 0$ where we take the cutoff to vanish, we have

$$\varepsilon_0(L;\alpha) = \frac{\pi}{8L^2}\frac{1}{\sinh^2\left(\frac{\alpha}{2L}\right)} = \frac{\pi}{2\alpha^2} - \frac{\pi}{24L^2} + \mathcal{O}\left(\alpha^2\right)\ , \tag{2.20}$$

so we have a divergent term in addition to a finite term, and then some terms that vanish when we remove the cutoff.

We are now in position to consider the difference between the vacuum energy with and without the presence of the plates. At fixed regulator α, the free-space reference is obtained from the $L \to \infty$ limit,

$$\begin{aligned}\Delta\varepsilon(L) = \varepsilon_0(L) - \varepsilon_0^{free} \to \Delta\varepsilon_{renorm}(L) &= \lim_{\alpha\to 0}\left[\varepsilon_0(L;\alpha) - \lim_{L\to\infty}\varepsilon_0(L;\alpha)\right] \\ &= \lim_{\alpha\to 0}\left[\frac{\pi}{2\alpha^2} - \frac{\pi}{24L^2} + \ldots - \frac{\pi}{2\alpha^2}\right] \\ &= -\frac{\pi}{24L^2}\end{aligned} \tag{2.21}$$

Thus the zero-point energy in the presence of plates at $x = 0, L$ is lower by a finite amount than in the free case without the plates. Taking the free-space energy as our zero of energy, the renormalised energy density between the boundaries is therefore finite and negative.

The vacuum energy between the plates is a form of potential energy which depends on the distance between the plates, and the gradient of the potential energy gives a force. The negative vacuum energy therefore leads to a force between the plates. For the $1+1$ dimensional scalar model considered here, the Casimir force between the plates is

$$F = -\frac{d}{dL}\Delta E = -\frac{d}{dL}\left(L\Delta\varepsilon_{renorm}\right) = -\frac{\pi}{24L^2} \tag{2.22}$$

We see a negative force; thus, the plates are pulled together. The Casimir force can be observed in the laboratory by placing two conducting plates in a vacuum chamber. For the physical electromagnetic field in $3+1$ dimensions, the numerical coefficient and power of L are different; for two ideal parallel conducting plates one finds $F/A = -\pi^2/(240L^4)$. The observation of the Casimir force illustrates that the vacuum energy contributed by virtual particles is a real physical phenomenon, and should not be taken lightly.

In order to find the physical vacuum energy in curved space (in general relativity in the presence of a non-trivial gravitational field), we will similarly compute the physical renormalised vacuum energy by subtracting flat space vacuum energy as we did with the Casimir effect above.

We will later see that in a Friedmann-Robertson-Walker (FRW) space-time the vacuum energy of a scalar field is like in flat space. From this argument, we would expect the renormalised cosmological constant to vanish. However, we do observe a tiny cosmological constant

$$\rho_\Lambda = \frac{\Lambda}{8\pi G_N} \sim M_{Pl}^2 H_0^2 \sim 10^{-120} M_{Pl}^4 \tag{2.23}$$

up to convention-dependent factors in the definition of the Planck mass. This is roughly 120 orders of magnitude below the naive Planck-scale estimate M_{Pl}^4. The observed acceleration may be due to a true cosmological constant, or it may instead arise from some dynamical form of dark energy.

But what is the physical interpretation of what is going on in the mathematically defined regularisation and renormalisation procedure? We can interpret the subtracting off the UV divergence as a shorthand for not knowing the theory above the UV cutoff, or the UV finite completion of the QFT. We can think that we are subtracting off a negative contribution to the vacuum energy contributed by the unknown physics above the cutoff.

String theory is a leading candidate for a UV completion of quantum gravity, although it is not the only example of a UV-complete quantum field-theoretic framework. In a globally supersymmetric theory with unbroken supersymmetry and a Minkowski vacuum, bosonic and fermionic zero-point contributions cancel and the vacuum energy vanishes. In supergravity and string theory, the situation is more subtle: supersymmetric vacua can, for example, have a negative cosmological constant. Since supersymmetry is not observed at experimentally accessible energies, it must be broken in our vacuum. After supersymmetry breaking, the cancellation of vacuum energies is generically lost, and contributions of order the supersymmetry-breaking scale to the fourth power naively arise. Thus a UV completion by itself does not automatically explain the extremely small observed cosmological constant.

This is the cosmological constant problem. The cosmological constant is very sensitive to vacuum-energy contributions from physics at many different scales. One can think of the renormalised vacuum energy density schematically as:

$$\rho_{\Lambda,renorm} = \rho_{\Lambda,bare} + \rho_{\Lambda,C.T.} + \rho_{\Lambda,quantum} \tag{2.24}$$

where the counterterm removes the regulator dependence and $\rho_{\Lambda,quantum}$ denotes the quantum contributions from the fields. With a hard cutoff Λ_{UV}, individual contributions can scale as Λ_{UV}^4; if the relevant scale is taken to be M_{Pl}, obtaining the observed value requires an enormous cancellation among the terms. Moreover, phase transitions and symmetry breaking can shift the vacuum energy by large amounts, so the small observed value must remain stable after such contributions are included.

This extraordinary sensitivity is the essence of the cosmological constant naturalness problem. Ideas for addressing it include dynamical mechanisms, modifications of the gravitational sector, and the string landscape together with anthropic selection arguments.

Quantum Fields in Curved Spacetime

Let us consider the scalar field Lagrangian density in curved spacetime

$$\mathcal{L}(x) = -\frac{1}{2}\sqrt{-g(\mathbf{x})}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(m^2 - \xi\mathcal{R}\right)\phi^2\right] \tag{3.1}$$

where ξ is the non-minimal coupling, and we have the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad \mu,\nu = 0,1,\ldots,(d-1) \tag{3.2}$$

with determinant

$$g(x) \equiv \det\left(g_{\mu\nu}\right) \tag{3.3}$$

Among local terms that are quadratic in ϕ, linear in the curvature, and require no additional mass scale, the term $\xi\mathcal{R}\phi^2$ is the unique non-minimal scalar-curvature coupling. Couplings involving additional powers of the curvature or additional derivatives are higher-dimensional operators and require appropriate inverse powers of a mass scale.

From the action

$$S = \int d^n\mathbf{x}\mathcal{L} \tag{3.4}$$

we can derive the equation of motion

$$\left(\Box + m^2 - \xi\mathcal{R}(\mathbf{x})\right)\phi(\mathbf{x}) = 0 \tag{3.5}$$

where the covariant d'Alembertian, with our sign convention, is

$$\Box\phi = -g^{\mu\nu}D_\mu D_\nu\phi = -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) \tag{3.6}$$

Thus in flat space with the $(-+++)$ metric convention, $\Box = \partial_t^2 - \nabla^2$.

We notice that on the covariant form in (3.5), with $\xi = 0$ it has the same form as in flat space, and so we say the field is minimally coupled in that case. On the other hand, we say the field has a conformal coupling for

$$\xi = \frac{1}{4}\frac{n-2}{n-1}, \tag{3.7}$$

because the equations of motion of a conformally coupled scalar field with $m = 0$ are conformally covariant under conformal transformations of the form

$$g_{\mu\nu}(\mathbf{x}) \to \bar{g}_{\mu\nu}(\mathbf{x}) = \Omega^2(\mathbf{x})\, g_{\mu\nu}(\mathbf{x}), \qquad \phi(\mathbf{x}) \to \bar{\phi}(\mathbf{x}) = \Omega^{-\frac{n-2}{2}}(\mathbf{x})\, \phi(\mathbf{x}). \quad (3.8)$$

To quantize the scalar field, learning from the treatment of the canonical quantization procedure in the first lecture, we want to find an orthonormal complete basis for our solutions to the equation of motion. To talk about orthonormality, we first need to have a generalized covariant definition of an inner product, with respect to which we can say the solutions are orthonormal. Assuming that the space-time is globally hyperbolic, let Σ be a space-like Cauchy hypersurface. We will, therefore, define the inner product

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma^\mu\, \phi_1(\mathbf{x})\, \overleftrightarrow{D}_\mu \phi_2^*(\mathbf{x}) \quad (3.9)$$

with

$$d\Sigma^\mu = n^\mu d\Sigma, \qquad d\Sigma = \sqrt{h}\, d^{n-1}x \quad (3.10)$$

where n^μ is the future-directed unit vector orthogonal to the space-like hypersurface Σ, $n^\mu n_\mu = -1$, and h is the determinant of the induced spatial metric on Σ. Furthermore,

$$\phi_1 \overleftrightarrow{D}_\mu \phi_2^* \equiv \phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1. \quad (3.11)$$

For scalar fields, $D_\mu \phi = \partial_\mu \phi$. If ϕ_1 and ϕ_2 both satisfy the Klein–Gordon equation, the Klein–Gordon current is covariantly conserved,

$$D_\mu \left(-i \phi_1 \overleftrightarrow{D}^\mu \phi_2^* \right) = 0. \quad (3.12)$$

Using Gauss' theorem and assuming that there is no flux through the boundary at spatial infinity, this implies that the inner product is independent of the choice of Cauchy hypersurface Σ. In flat space, the inner product just reduces to

$$(\phi_1, \phi_2) = -i \int_t \phi_1 \overleftrightarrow{\partial}_t \phi_2^* d^{n-1}\boldsymbol{x} \quad (3.13)$$

There exists a complete set of orthonormal modes, which are solutions to the equation of motion from (3.5)

$$(u_i, u_j) = \delta_{ij}, \quad \left(u_i^*, u_j^*\right) = -\delta_{ij}, \quad \left(u_i, u_j^*\right) = 0 \quad (3.14)$$

The indices cover the relevant set of labels, which will become clearer when we consider a specific example in the next section. These are the Klein–Gordon normalisation conditions. When the mode equation reduces to an ordinary differential equation in time, the corresponding normalisation condition is often referred to as a Wronskian condition.

For example, in Minkowski space the normalised positive-frequency modes may be chosen as

$$u_{\boldsymbol{k}}\left(t, \boldsymbol{x}\right) = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}\left(2\pi\right)^{n-1}}} e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.15}$$

with the complex conjugate modes having negative Klein–Gordon norm. For continuum labels the Kronecker delta in the normalization conditions is replaced by a Dirac delta.

We can now write a general field, which is a solution to the equation of motion, as an expansion in the orthonormal basis mode functions

$$\phi\left(\mathbf{x}\right) = \sum_i \left[a_i u_i\left(\mathbf{x}\right) + a_i^+ u_i^*\left(\mathbf{x}\right)\right] \tag{3.16}$$

3.1 Covariant Quantization

In order to quantize the field, we can now follow the usual procedure of canonical quantization and impose the commutation relations

$$\left[a_i, a_j^+\right] = \delta_{ij}, \qquad \left[a_i, a_j\right] = \left[a_i^+, a_j^+\right] = 0 \ , \tag{3.17}$$

and we can construct a Fock space by following a similar approach to how we did it in Minkowski space.

There are, however, some important differences. In Minkowski space, there is a global timelike Killing vector corresponding to time translations. In inertial coordinates this Killing vector is $\frac{\partial}{\partial t}$ and is orthogonal to the space-like hypersurfaces with $t =$ constant. The Minkowski mode functions are eigenfunctions of $\frac{\partial}{\partial t}$ with eigenvalues $-i\omega$. Spatial translations provide the conserved spatial momentum, while rotations and boosts complete the Poincaré symmetries. The Minkowski vacuum is selected as the Poincaré-invariant state satisfying the usual positive-energy condition. This provides a preferred vacuum in Minkowski space.

However, in curved space, there is no Poincaré invariance, and there is no unambiguous definition of the vacuum in general. Generally in curved space, there might be no Killing vectors. If a suitable globally defined timelike Killing vector exists, it may provide a preferred notion of positive frequency and hence a preferred vacuum, although global structure, horizons, and boundary conditions can introduce further subtleties.

One way to think about it is that a generic curved space-time lacks a preferred time-translation symmetry with respect to which positive and negative frequency can be defined. Consequently, there is, in general, no unique natural decomposition of the space of solutions into positive- and negative-frequency modes, and hence no unique particle or vacuum notion.

Let us therefore compare two different complete orthonormal sets $u_i\left(\mathbf{x}\right)$ and $\bar{u}_j\left(\mathbf{x}\right)$ and the field expansion, which can be written equivalently in both bases

$$\begin{aligned}\phi\left(\mathbf{x}\right) &= \sum_i \left[a_i u_i\left(\mathbf{x}\right) + a_i^+ u_i^*\left(\mathbf{x}\right)\right] \\ \phi\left(\mathbf{x}\right) &= \sum_j \left[\bar{a}_j \bar{u}_j\left(\mathbf{x}\right) + \bar{a}_j^+ \bar{u}_j^*\left(\mathbf{x}\right)\right]\end{aligned} \tag{3.18}$$

Defining the vacuum state in the usual way, we are led to two different definitions of the vacuum

$$a_i|0\rangle = 0, \qquad \bar{a}_j|\bar{0}\rangle = 0 \tag{3.19}$$

and therefore two different particle and vacuum definitions.

To understand the difference between these two definitions of the vacuum, let us expand the mode functions in each other's basis

$$\begin{aligned}\bar{u}_j &= \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right) \\ u_i &= \sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right) \ .\end{aligned} \tag{3.20}$$

These transformations of the mode functions between the two bases are called Bogoliubov transformations, with the Bogoliubov coefficients being α_{ij} and β_{ij}. Using the orthonormality of the mode functions, we have that the Bogoliubov coefficients are given by

$$\alpha_{ij} = \left(\bar{u}_i, u_j\right), \quad \beta_{ij} = -\left(\bar{u}_i, u_j^*\right) \ . \tag{3.21}$$

Bogoliubov gymnastics

To prove (3.1), we start from the Bogoliubov transformation

$$\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right) , \tag{3.22}$$

where the first index labels the $\bar{u}_j$ basis and the second the u_i basis. Taking the complex conjugate gives

$$\bar{u}_j^* = \sum_i \left(\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i\right) . \tag{3.23}$$

Before deriving the inverse transformation, let us note that orthonormality and completeness imply the Bogoliubov identities

$$\sum_j \left(\alpha_{ji}^* \alpha_{jk} - \beta_{ji} \beta_{jk}^*\right) = \delta_{ik}, \qquad \sum_j \left(\alpha_{ji}^* \beta_{jk} - \beta_{ji} \alpha_{jk}^*\right) = 0. \tag{3.24}$$

These are the inverse-form identities; their relation to the more familiar identities with the barred indices contracted is shown below.

Now consider the combination

$$\sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right) . \tag{3.25}$$

Substituting the expansions for $\bar{u}_j$ and $\bar{u}_j^*$ gives

$$\begin{aligned}
&\sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right) &(3.26)\\
&= \sum_j \alpha_{ji}^* \sum_k \left(\alpha_{jk} u_k + \beta_{jk} u_k^*\right) - \sum_j \beta_{ji} \sum_k \left(\alpha_{jk}^* u_k^* + \beta_{jk}^* u_k\right) &(3.27)\\
&= \sum_k \left[\sum_j \left(\alpha_{ji}^* \alpha_{jk} - \beta_{ji} \beta_{jk}^*\right)\right] u_k + \sum_k \left[\sum_j \left(\alpha_{ji}^* \beta_{jk} - \beta_{ji} \alpha_{jk}^*\right)\right] u_k^*. &(3.28)
\end{aligned}$$

Using the Bogoliubov identities

$$\sum_j \left(\alpha_{ji}^* \alpha_{jk} - \beta_{ji} \beta_{jk}^*\right) = \delta_{ik}, \qquad \sum_j \left(\alpha_{ji}^* \beta_{jk} - \beta_{ji} \alpha_{jk}^*\right) = 0, \tag{3.29}$$

we obtain

$$\sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right) = \sum_k \delta_{ik} u_k = u_i. \tag{3.30}$$

Hence the inverse transformation is

$$u_i = \sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \right). \tag{3.31}$$

To derive the Bogoliubov identities, assume that both sets of mode functions $\{u_i, u_i^*\}$ and $\{\bar{u}_j, \bar{u}_j^*\}$ are complete orthonormal sets with respect to the Klein–Gordon inner product $(\cdot, \cdot)$, satisfying

$$(u_i, u_k) = \delta_{ik}, \qquad (u_i^*, u_k^*) = -\delta_{ik}, \qquad (u_i, u_k^*) = 0, \tag{3.32}$$

and similarly

$$(\bar{u}_j, \bar{u}_l) = \delta_{jl}, \qquad (\bar{u}_j^*, \bar{u}_l^*) = -\delta_{jl}, \qquad (\bar{u}_j, \bar{u}_l^*) = 0. \tag{3.33}$$

The Bogoliubov transformation is

$$\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right). \tag{3.34}$$

Its complex conjugate is

$$\bar{u}_j^* = \sum_i \left(\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i \right). \tag{3.35}$$

First identity.

$$(\bar{u}_j, \bar{u}_l) = \sum_i \left(\alpha_{ji} \alpha_{li}^* - \beta_{ji} \beta_{li}^* \right) = \delta_{jl}. \tag{3.36}$$

Second identity.

$$(\bar{u}_j, \bar{u}_l^*) = \sum_i \left(\alpha_{ji} \beta_{li} - \beta_{ji} \alpha_{li} \right) = 0. \tag{3.37}$$

Equivalent inverse form. Because the transformation relates two complete bases, it is invertible. The two identities above state that the Bogoliubov transformation preserves the Klein–Gordon inner product. The inverse transformation must therefore preserve the inner product as well, which gives

$$\sum_j \left(\alpha_{ji}^* \alpha_{jk} - \beta_{ji} \beta_{jk}^* \right) = \delta_{ik}, \qquad \sum_j \left(\alpha_{ji}^* \beta_{jk} - \beta_{ji} \alpha_{jk}^* \right) = 0. \tag{3.38}$$

These are precisely the identities used above to obtain the inverse transformation.

Now equating the two field expansions

$$\sum_i \left[a_i u_i(\mathbf{x}) + a_i^+ u_i^*(\mathbf{x})\right] = \sum_i \left[\bar{a}_i \bar{u}_i(\mathbf{x}) + \bar{a}_i^+ \bar{u}_i^*(\mathbf{x})\right] \tag{3.39}$$

and taking the inner product with u_i on both sides of the equality, we obtain

$$a_i = \sum_j \left[\bar{a}_j(\bar{u}_j, u_i) + \bar{a}_j^+\left(\bar{u}_j^*, u_i\right)\right] . \tag{3.40}$$

Using the scalar product from (3.14) we get an operator

$$a_i = \sum_j \left[\alpha_{ji}\bar{a}_j + \beta_{ji}^*\bar{a}_j^+\right] \tag{3.41}$$

Similarly for the other orthonormal set, we have:

$$\bar{a}_j = \sum_i \left[\alpha_{ji}^* a_i - \beta_{ji}^* a_i^+\right] \tag{3.42}$$

It is now clear that if $\beta_{ij} \neq 0$, then the two definitions of particles and vacuum are different, and the vacuum from $\bar{a}_j|\bar{0}\rangle = 0$ will not be annihilated by a_i

$$a_i|\bar{0}\rangle = \sum_j \beta_{ji}^*|\bar{1}_j\rangle \neq 0 \tag{3.43}$$

In fact, the number operator $N_i = a_i^+ a_i$ gives

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 , \tag{3.44}$$

which means that $|\bar{0}\rangle$ vacuum contains particles of the u_i mode. Thus, the $|\bar{0}\rangle$ vacuum is an excited state in the Fock space defined from the $|0\rangle$ vacuum state. For infinitely many degrees of freedom, the two Fock representations are unitarily equivalent only when the Bogoliubov coefficients satisfy the appropriate convergence condition, schematically $\sum_{ij} |\beta_{ij}|^2 < \infty$. Otherwise, the two representations can be unitarily inequivalent.

This Bogoliubov mixing of positive- and negative-frequency modes is the basis of the mechanism whereby the vacuum changes in time in an expanding universe, and particle production gives us a generation of structures from initial quantum fluctuations during inflation. It is also the basis of the Unruh effect, where an accelerated observer observes particles in the Minkowski vacuum, an effect similar to Hawking radiation where an observer at rest far away from a black hole observes particles or radiation from the black hole. We will discuss these examples in more detail in the next sections.

3.2 Killing Vectors

Consider a vector field defined by an integral curve $x^\mu(t)$. One can think of $x^\mu(t)$ as mapping the space-time coordinates of an observer as a function of time. Then we can think of the vector

$$\frac{dx^\mu}{dt} = V^\mu \tag{3.45}$$

and define the Lie derivative of a scalar, vector, or tensor along that vector field

$$\begin{aligned} \text{Scalar:} \quad & \mathscr{L}_V f = V^\mu \partial_\mu f \\ \text{Vector:} \quad & \mathscr{L}_V U^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \end{aligned} \tag{3.46}$$

The Lie derivative of the metric is

$$\mathscr{L}_V g_{\mu\nu} = V^\sigma D_\sigma g_{\mu\nu} + \left(D_\mu V^\lambda\right) g_{\lambda\nu} + \left(D_\nu V^\lambda\right) g_{\mu\lambda} \tag{3.47}$$

Since the covariant derivative of the metric vanishes, $D_\sigma g_{\mu\nu} = 0$, it is equivalent to

$$\mathscr{L}_V g_{\mu\nu} = D_\mu V_\nu + D_\nu V_\mu. \tag{3.48}$$

If we have a displacement in the direction given by a vector X^μ that leaves all distance relationships unchanged, then there is an isometry, and the vector field is called a Killing vector.

More precisely, consider the infinitesimal diffeomorphism generated by X^μ,

$$x'^\mu = x^\mu + \epsilon X^\mu. \tag{3.49}$$

By definition, the metric transforms as

$$g'_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\, g_{\rho\sigma}(x'). \tag{3.50}$$

For an isometry we require $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$. We have

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \epsilon\,\partial_{\nu} X^{\mu} \tag{3.51}$$

such that

$$g'_{\mu\nu}(x) = (\delta^{\rho}_{\mu} + \epsilon\,\partial_{\mu} X^{\rho})(\delta^{\sigma}_{\nu} + \epsilon\,\partial_{\nu} X^{\sigma})\,g_{\rho\sigma}(x^{\lambda} + \epsilon X^{\lambda}) \tag{3.52}$$

$$= (\delta^{\rho}_{\mu} + \epsilon\,\partial_{\mu} X^{\rho})(\delta^{\sigma}_{\nu} + \epsilon\,\partial_{\nu} X^{\sigma})\Big[g_{\rho\sigma}(x) + \epsilon X^{\lambda}\partial_{\lambda} g_{\rho\sigma}(x) + \dots\Big] \tag{3.53}$$

$$= g_{\mu\nu}(x) + \epsilon\Big[g_{\mu\sigma}\,\partial_{\nu} X^{\sigma} + g_{\nu\sigma}\,\partial_{\mu} X^{\sigma} + X^{\lambda}\partial_{\lambda} g_{\mu\nu}\Big] + O(\epsilon^2) \tag{3.54}$$

$$= g_{\mu\nu} + \epsilon(\mathscr{L}_X g)_{\mu\nu} \tag{3.55}$$

$$= g_{\mu\nu} + \epsilon\big(D_{\mu} X_{\nu} + D_{\nu} X_{\mu}\big) + O(\epsilon^2). \tag{3.56}$$

Imposing the isometry condition $g'_{\mu\nu} = g_{\mu\nu}$ therefore gives

$$(\mathscr{L}_X g)_{\mu\nu} = D_{\mu} X_{\nu} + D_{\nu} X_{\mu} = 0.$$

For a Killing vector X^{μ}, this gives the Killing equation

$$D_{\mu} X_{\nu} + D_{\nu} X_{\mu} = 0.$$

If some V^{μ} is a Killing vector, we then have by definition

$$\mathscr{L}_V g_{\mu\nu} = 2D_{(\mu} V_{\nu)} = 0 \tag{3.57}$$

and locally one can choose coordinates adapted to the Killing flow such that V^{μ} points along one coordinate direction and the metric is independent of that coordinate.

The Killing vectors imply conserved quantities. For example, if u^{μ} is the tangent vector to a geodesic and V^{μ} is a Killing vector, then

$$\frac{d}{d\tau}\,(V_{\mu} u^{\mu}) = 0. \tag{3.58}$$

Thus a timelike Killing vector gives a conserved energy along the geodesic, while spatial or rotational Killing vectors give conserved momentum or angular momentum.

From the covariant conservation of the energy-momentum tensor $D_{\mu} T^{\mu\nu} = 0$, it also follows that the current

$$J^{\mu} = V_{\nu} T^{\mu\nu} \tag{3.59}$$

is conserved. Indeed, assuming the stress-energy tensor is symmetric,

$$D_\mu J^\mu = (D_\mu V_\nu)T^{\mu\nu} + V_\nu D_\mu T^{\mu\nu} = T^{\mu\nu} D_{(\mu}V_{\nu)} = 0. \tag{3.60}$$

Equivalently,

$$D_\mu(V_\nu T^{\mu\nu}) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}V_\nu T^{\mu\nu}) = 0\ . \tag{3.61}$$

Provided there is no flux through the boundary, this gives a conserved charge obtained by integrating J^μ over a space-like hypersurface.

Scalar Field in FRW Spacetime

On very large scales, scales larger than the scales of galaxy clusters, the universe appears homogeneous and isotropic. Furthermore, the measured spatial curvature of the universe is consistent with zero. With these assumptions, the most general space-time metric compatible with homogeneity, isotropy, and flat spatial sections is the spatially flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] \tag{4.1}$$

Where $a(t)$ is the scale factor.

In this section, we will assume that the scalar field contributes a subdominant energy density, and that we can ignore its influence on the geometry. We will therefore treat the metric as fixed and disregard the influence of the fields on the geometry.

Now consider the minimally coupled, free, real massive scalar field $\phi(x)$

$$S = -\int d^4\mathbf{x}\sqrt{-g}\left[\frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha\phi\right)\left(\partial_\beta\phi\right) + \frac{1}{2}m^2\phi^2\right] . \tag{4.2}$$

with the equation of motion as before

$$\Box_g\phi + m^2\phi = 0 . \tag{4.3}$$

with $\Box_g = -g^{\mu\nu}D_\mu D_\nu$.

In FRW space-time, it is convenient to introduce conformal time

$$\eta(t) \equiv \int_{t_0}^{t}\frac{dt}{a(t)} \qquad \text{with} \qquad a d\eta = dt \tag{4.4}$$

When differentiating with respect to conformal time, we denote it with a prime, like $\partial_\eta\phi = \phi'$. In the coordinates $(\eta, \boldsymbol{x})$, the infinitesimal line interval becomes

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\boldsymbol{x}^2\right] . \tag{4.5}$$

This means that the metric is conformally flat

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = a^2(\eta)\eta_{\mu\nu}, \qquad g^{\mu\nu}(\eta, \boldsymbol{x}) = a^{-2}(\eta)\eta^{\mu\nu} \tag{4.6}$$

and related to the Minkowski metric by a conformal transformation of the form $g_{\mu\nu} \to \Omega^2(\mathbf{x})\, g_{\mu\nu}$, or equivalently $g^{\mu\nu} \to \Omega^{-2}(\mathbf{x})\, g^{\mu\nu}$, preserving angles and ratios of magnitudes.

To solve the equation of motion, it is also convenient to make a field redefinition

$$\chi \equiv a(\eta)\,\phi \ . \tag{4.7}$$

Then using $\sqrt{-g} = a^4$ we have

$$\sqrt{-g}m^2\phi^2 = m^2 a^2 \chi^2 \tag{4.8}$$

and

$$\sqrt{-g}g^{\alpha\beta}\left(\partial_\alpha\phi\right)\left(\partial_\beta\phi\right) = a^2\left(-\phi'^2 + \left(\nabla\phi\right)^2\right) \tag{4.9}$$

with

$$a^2\phi'^2 = \chi'^2 + \chi^2\frac{a''}{a} - \left[\chi^2\frac{a'}{a}\right]' \tag{4.10}$$

where the last term is a total derivative and can be dropped under the integral in the action

$$S = \frac{1}{2}\int d^3\boldsymbol{x}d\eta\left[\chi'^2 - \left(\nabla\chi\right)^2 - m_{eff}^2(\eta)\,\chi^2\right] \tag{4.11}$$

with

$$m_{eff}^2(\eta) \equiv m^2a^2 - \frac{a''}{a} \ . \tag{4.12}$$

The action (4.11) is identical to the action of a scalar field in Minkowski space with a time-dependent effective mass. Thus, all the effects of gravity are encapsulated in the time dependence of m_{eff} for this free, minimally coupled scalar field after the field redefinition $\chi = a\phi$.

The action is explicitly time-dependent, which will also be true for the related Hamiltonian, and so the energy of χ is not conserved. This leads to the phenomenon of particle creation in the external field of gravity (with the energy supplied by the gravitational field).

4.1 Mode Expansion

The equation of motion for the field, $\chi(\eta, \boldsymbol{x})$, is

$$\chi'' - \nabla^2\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \ . \tag{4.13}$$

We notice that the Laplacian ∇^2 is the usual one from flat space, having $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ as eigenfunctions (a consequence of vanishing spatial curvature), and we can expand in Fourier modes

$$\chi(\eta, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \chi_{\boldsymbol{k}}(\eta)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.14}$$

Inserting the Fourier expansion into the equation of motion, we obtain the Fourier mode equation

$$\chi_{\boldsymbol{k}}'' + \left[k^2 + m^2 a^2(\eta) - \frac{a''}{a}\right]\chi_{\boldsymbol{k}} = 0\ , \tag{4.15}$$

which we can write in the form of a harmonic oscillator

$$\chi_{\boldsymbol{k}}'' + \omega_{\boldsymbol{k}}^2(\eta)\,\chi_{\boldsymbol{k}} = 0 \tag{4.16}$$

with a time dependent frequency $\omega_k^2 \equiv k^2 + m_{eff}^2(\eta)$.

In order to find a general solution, let us first choose one complex solution, a mode function $V_{\boldsymbol{k}}(\eta)$ such that

$$V_{\boldsymbol{k}}'' + \omega_{\boldsymbol{k}}^2(\eta)\, V_{\boldsymbol{k}} = 0 \tag{4.17}$$

with non-vanishing Wronskian $W[V_{\boldsymbol{k}}, V_{\boldsymbol{k}}^*] \neq 0$. Since $\omega_{\boldsymbol{k}}^2$ is real, $V_{\boldsymbol{k}}^*$ is then also a solution and is linearly independent of $V_{\boldsymbol{k}}$.

Then the general solution, $\chi_{\boldsymbol{k}}(\eta)$, can be written as a linear combination of $V_{\boldsymbol{k}}$ and $V_{\boldsymbol{k}}^*$ as:

$$\chi_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}^- V_{\boldsymbol{k}}^*(\eta) + a_{-\boldsymbol{k}}^+ V_{\boldsymbol{k}}(\eta)\right] \tag{4.18}$$

If χ is real we have

$$\chi_{\boldsymbol{k}}^* = \chi_{-\boldsymbol{k}} \quad \Rightarrow \quad a_{\boldsymbol{k}}^+ = \left(a_{\boldsymbol{k}}^-\right)^*\ , \tag{4.19}$$

and we obtain the following mode expansion

$$\chi(\eta, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}^- V_{\boldsymbol{k}}^*(\eta) + a_{-\boldsymbol{k}}^+ V_{\boldsymbol{k}}(\eta)\right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.20}$$

Since $\omega_{\boldsymbol{k}}$ depends only on $k = |\boldsymbol{k}|$, we may choose $V_{-\boldsymbol{k}} = V_{\boldsymbol{k}}$. Changing $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term then gives the equivalent form

$$\chi(\eta, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}^- V_{\boldsymbol{k}}^*(\eta)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a_{\boldsymbol{k}}^+ V_{\boldsymbol{k}}(\eta)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\ . \tag{4.21}$$

One can now verify that

$$a_{\boldsymbol{k}}^{-} = \sqrt{2}\frac{V_{\boldsymbol{k}}'\chi_{\boldsymbol{k}} - V_{\boldsymbol{k}}\chi_{\boldsymbol{k}}'}{V_{\boldsymbol{k}}'V_{\boldsymbol{k}}^{*} - V_{\boldsymbol{k}}V_{\boldsymbol{k}}^{*\prime}} \equiv \sqrt{2}\frac{W\left[V_{\boldsymbol{k}}, \chi_{\boldsymbol{k}}\right]}{W\left[V_{\boldsymbol{k}}, V_{\boldsymbol{k}}^{*}\right]} \ , \tag{4.22}$$

where W is the Wronskian, and again we have $a_{\boldsymbol{k}}^{+} = \left(a_{\boldsymbol{k}}^{-}\right)^{*}$.

Thus, having obtained the mode expansion, we are ready to quantise.

4.2 Quantization of a Scalar Field in FRW

The mode expansion for the field operator $\hat{\chi}$ is obtained by promoting constants to operators and taking

$$a_{\boldsymbol{k}}^{\pm} \to \hat{a}_{\boldsymbol{k}}^{\pm} \ , \tag{4.23}$$

and we get the mode expansion for the field operator

$$\hat{\chi}\left(\eta, \boldsymbol{x}\right) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2}}\left(V_{\boldsymbol{k}}^{*}\left(\eta\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}}^{-} + V_{\boldsymbol{k}}\left(\eta\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}}^{+}\right) \tag{4.24}$$

where we impose the usual commutators

$$\left[\hat{a}_{\boldsymbol{k}}^{-}, \hat{a}_{\boldsymbol{k}'}^{+}\right] = \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right), \quad \left[\hat{a}_{\boldsymbol{k}}^{-}, \hat{a}_{\boldsymbol{k}'}^{-}\right] = \left[\hat{a}_{\boldsymbol{k}}^{+}, \hat{a}_{\boldsymbol{k}'}^{+}\right] = 0 \tag{4.25}$$

consistent with

$$\left[\hat{\chi}\left(\eta, \boldsymbol{x}_1\right), \hat{\chi}'\left(\eta, \boldsymbol{x}_2\right)\right] = i\delta^{(3)}\left(\boldsymbol{x}_1 - \boldsymbol{x}_2\right) \ , \tag{4.26}$$

when the Wronskian condition is satisfied

$$\mathrm{Im}\left(V_{\boldsymbol{k}}'V_{\boldsymbol{k}}^{*}\right) = \frac{V_{\boldsymbol{k}}'V_{\boldsymbol{k}}^{*} - V_{\boldsymbol{k}}V_{\boldsymbol{k}}^{*\prime}}{2i} = \frac{W\left[V_{\boldsymbol{k}}, V_{\boldsymbol{k}}^{*}\right]}{2i} = 1 \ . \tag{4.27}$$

Equivalently, $W\left[V_{\boldsymbol{k}}, V_{\boldsymbol{k}}^{*}\right] = 2i$. With the factor $1/\sqrt{2}$ used in the mode expansion, this is precisely the Klein–Gordon normalisation condition discussed in the previous lecture.

Choice of Vacuum

Remember we defined the vacuum state as the state annihilated by the annihilation operator

$$\hat{a}_{\boldsymbol{k}}^{-}|0\rangle = 0 \ . \tag{5.1}$$

But we have seen that this definition is not unique and $\hat{a}_{\boldsymbol{k}}^{-}$ is fixed by the choice of mode function $V_{\boldsymbol{k}}(\eta)$. Selecting a specific vacuum, therefore, comes down to selecting a preferred mode function. If we expand the field in a different set of mode functions, the vacuum defined by the related annihilation operators changes.

Since the vacuum is not defined in a mathematically unique way, the choice of a vacuum state is a physical condition that needs to be imposed. There are different physical conditions one can impose and, therefore, different ways to choose the vacuum state, depending on the physics one is describing. Different observers may naturally use different notions of positive frequency and hence different particle definitions. For example, static and freely falling observers near a black-hole horizon, or cosmological observers at different times in the universe, need not agree on the particle content of a given quantum state.

5.1 Instantaneous Vacuum

In flat space, we defined the vacuum as the eigenstate with the lowest energy. However, now the Hamiltonian depends explicitly on time, so there is no time-independent lowest energy state that could serve as the vacuum.

Instead, we can try to define the vacuum to be the lowest energy eigenstate at some particular moment $\eta = \eta_0$ of the instantaneous Hamiltonian $H(\eta_0)$ at that moment. The Hamiltonian of the rescaled canonical field is

$$\hat{H}\left(\eta\right) = \frac{1}{2}\int d^3\boldsymbol{x}\,\left[\hat{\pi}^2 + \left(\nabla\hat{\chi}\right)^2 + m_{eff}^2\left(\eta\right)\hat{\chi}^2\right] \tag{5.2}$$

where

$$\hat{\pi} = \frac{\partial\mathcal{L}}{\partial\hat{\chi}'} = \hat{\chi}' \ . \tag{5.3}$$

Each Fourier mode therefore behaves as a harmonic oscillator with instantaneous frequency $\omega_{\boldsymbol{k}}^2(\eta) = k^2 + m_{eff}^2(\eta)$.

Using the mode expansion for χ (see end of lecture four) gives

$$E\left(\eta\right) \equiv \langle 0|\hat{H}\left(\eta\right)|0\rangle = \int \frac{d^3\boldsymbol{k}}{\left(2\pi\right)^3}\frac{\left|V_{\boldsymbol{k}}'\left(\eta\right)\right|^2 + \omega_{\boldsymbol{k}}^2\left(\eta\right)\left|V_{\boldsymbol{k}}\left(\eta\right)\right|^2}{4} \ . \tag{5.4}$$

In order to minimize the energy at $\eta = \eta_0$, we need to minimize

$$|V'_{\boldsymbol{k}}(\eta)|^2 + \omega^2_{\boldsymbol{k}}(\eta)\,|V_{\boldsymbol{k}}(\eta)|^2 \tag{5.5}$$

for every k-mode at $\eta = \eta_0$. Let us denote

$$V_{\boldsymbol{k}}(\eta_0) = s_{\boldsymbol{k}}, \quad V'_{\boldsymbol{k}}(\eta_0) = t_{\boldsymbol{k}} \tag{5.6}$$

where the Wronskian condition for the mode functions V_k implies

$$s^*_{\boldsymbol{k}} t_{\boldsymbol{k}} - s_{\boldsymbol{k}} t^*_{\boldsymbol{k}} = 2i\ . \tag{5.7}$$

Now, we need to minimize

$$|t_{\boldsymbol{k}}|^2 + \omega^2(\eta_0)\,|s_{\boldsymbol{k}}|^2\ . \tag{5.8}$$

The phase of the mode function is a free parameter, so we can without loss of generality take $s_{\boldsymbol{k}}$ to be real and write $t_{\boldsymbol{k}} = t_{1\boldsymbol{k}} + it_{2\boldsymbol{k}}$. From the Wronskian we then have

$$s_{\boldsymbol{k}} = \frac{2i}{t_{\boldsymbol{k}} - t^*_{\boldsymbol{k}}} = \frac{1}{t_{2\boldsymbol{k}}} \tag{5.9}$$

and inserting into (5.4)

$$4E(\eta_0) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[t^2_{1\boldsymbol{k}} + t^2_{2\boldsymbol{k}} + \frac{\omega^2_{\boldsymbol{k}}(\eta_0)}{t^2_{2\boldsymbol{k}}}\right] \tag{5.10}$$

For $\omega^2_{\boldsymbol{k}}(\eta_0) > 0$ there is a minimum for

$$t_{1\boldsymbol{k}} = 0, \qquad t_{2\boldsymbol{k}} = \sqrt{\omega_{\boldsymbol{k}}(\eta_0)} \tag{5.11}$$

while for $\omega^2_{\boldsymbol{k}}(\eta_0) \leq 0$ there is no normalizable instantaneous ground state of harmonic-oscillator type; for $\omega^2_{\boldsymbol{k}} < 0$ the instantaneous Hamiltonian is unbounded below, while for $\omega_{\boldsymbol{k}} = 0$ the mode behaves as a free particle.

For $\omega^2_{\boldsymbol{k}}(\eta_0) > 0$ the instantaneous vacuum at $\eta = \eta_0$ is given by

$$V_{\boldsymbol{k}}(\eta_0) = \frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\eta_0)}}, \qquad V'_{\boldsymbol{k}}(\eta_0) = i\sqrt{\omega_{\boldsymbol{k}}(\eta_0)} = i\omega_{\boldsymbol{k}}(\eta_0)\,V_{\boldsymbol{k}}(\eta_0)\ . \tag{5.12}$$

Remember that in Minkowski space-time $\omega_{\boldsymbol{k}}$ is time independent and yields the mode function

$$V_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} e^{i\omega_{\boldsymbol{k}}\eta} \tag{5.13}$$

which agrees with the instantaneous vacuum solution above (up to a phase).

However, in time-dependent backgrounds, the instantaneous vacuum at different times will generically differ: $|0\rangle_{\eta_0} \neq |0\rangle_{\eta_1}$. Suppose $V_{\boldsymbol{k}}(\eta)$ and $U_{\boldsymbol{k}}(\eta)$ are the mode functions describing the instantaneous vacuum at $\eta = \eta_0$ and $\eta = \eta_1$. After evolving both exact mode functions to a common time, for example $\eta = \eta_0$, we can relate them by Bogoliubov coefficients $\alpha_{\boldsymbol{k}}$ and $\beta_{\boldsymbol{k}}$:

$$\begin{aligned} V_{\boldsymbol{k}}^* (\eta_0) &= \alpha_{\boldsymbol{k}} U_{\boldsymbol{k}}^* (\eta_0) + \beta_{\boldsymbol{k}} U_{\boldsymbol{k}} (\eta_0) \\ V_{\boldsymbol{k}}^{*\prime} (\eta_0) &= \alpha_{\boldsymbol{k}} U_{\boldsymbol{k}}^{*\prime} (\eta_0) + \beta_{\boldsymbol{k}} U_{\boldsymbol{k}}^{\prime} (\eta_0) \end{aligned} \tag{5.14}$$

where the coefficients can be expressed as

$$\alpha_{\boldsymbol{k}} = \left. \frac{U_{\boldsymbol{k}}' V_{\boldsymbol{k}}^* - U_{\boldsymbol{k}} V_{\boldsymbol{k}}^{*\prime}}{2i} \right|_{\eta_0} , \qquad \beta_{\boldsymbol{k}}^* = \left. \frac{U_{\boldsymbol{k}}' V_{\boldsymbol{k}} - U_{\boldsymbol{k}} V_{\boldsymbol{k}}'}{2i} \right|_{\eta_0} . \tag{5.15}$$

The Wronskian normalization implies

$$|\alpha_{\boldsymbol{k}}|^2 - |\beta_{\boldsymbol{k}}|^2 = 1 . \tag{5.16}$$

We could also have used asymptotic values at $\eta \to -\infty$ instead of values at η_0, which would then appear as an excited state in the instantaneous vacuum at any finite time. We thus see how particles and matter in our universe can be created out of nothing in the infinite past.

5.2 Adiabatic Vacuum

If the effective frequency varies slowly with time, it is useful to consider the adiabatic vacuum. Again the mode equation is

$$V_{\boldsymbol{k}}'' + \omega_{\boldsymbol{k}}^2 (\eta) V_{\boldsymbol{k}} = 0, \qquad \omega_{\boldsymbol{k}}^2 (\eta) \equiv k^2 + m_{eff}^2 (\eta) \tag{5.17}$$

in a region where $\omega_{\boldsymbol{k}}^2 > 0$. Adiabaticity requires, schematically,

$$\frac{|\omega_{\boldsymbol{k}}'|}{\omega_{\boldsymbol{k}}^2} \ll 1, \qquad \frac{|\omega_{\boldsymbol{k}}''|}{\omega_{\boldsymbol{k}}^3} \ll 1, \quad \ldots \tag{5.18}$$

The mode equation possesses a formal WKB-type ansatz. With the normalization and phase convention used in Lecture 4, where $V_{\boldsymbol{k}}^*$ multiplies the annihilation operator and $W[V_{\boldsymbol{k}}, V_{\boldsymbol{k}}^*] = 2i$, we write

$$V_{\boldsymbol{k}} = \frac{1}{\sqrt{W_{\boldsymbol{k}}}} \exp\left[i \int^{\eta} W_{\boldsymbol{k}} (\eta') \, d\eta' \right] \tag{5.19}$$

where

$$W_{\boldsymbol{k}}^2(\eta) = \omega_{\boldsymbol{k}}^2(\eta) - \frac{1}{2}\left(\frac{W_{\boldsymbol{k}}''}{W_{\boldsymbol{k}}} - \frac{3}{2}\frac{W_{\boldsymbol{k}}'^2}{W_{\boldsymbol{k}}^2}\right) . \tag{5.20}$$

If $W_{\boldsymbol{k}}$ satisfies this nonlinear equation exactly, the ansatz is an exact solution. The adiabatic/WKB approximation is obtained by solving the equation iteratively in time derivatives of $\omega_{\boldsymbol{k}}$.

In the adiabatic limit, where the time variation is slow, we have to zeroth order

$$W_{\boldsymbol{k}} \approx W_{\boldsymbol{k}}^{(0)}(\eta) \equiv \omega_{\boldsymbol{k}}(\eta) \ , \tag{5.21}$$

so the zeroth order for the WKB-type solution becomes

$$V_{\boldsymbol{k}}^{(0)} = \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \exp\left[i\int_{\eta_0}^{\eta} \omega_{\boldsymbol{k}}(\eta')\, d\eta'\right] \tag{5.22}$$

This solution approximately satisfies the instantaneous minimum energy condition. To see this, we note that the mode functions of the zero'th order adiabatic vacuum are defined by:

$$V_{\boldsymbol{k}}(\eta_0) = V_{\boldsymbol{k}}^{(0)}(\eta_0), \qquad V_{\boldsymbol{k}}'(\eta)\big|_{\eta=\eta_0} = V_{\boldsymbol{k}}^{(0)\prime}(\eta)\Big|_{\eta=\eta_0} \tag{5.23}$$

which implies the normalization of the full mode function at $\eta = \eta_0$

$$V_{\boldsymbol{k}}(\eta_0) = \frac{1}{\sqrt{\omega_{\boldsymbol{k}}(\eta_0)}}, \qquad V_{\boldsymbol{k}}'(\eta)\big|_{\eta=\eta_0} = \left(i\omega_{\boldsymbol{k}} - \frac{1}{2}\frac{\omega_{\boldsymbol{k}}'}{\omega_{\boldsymbol{k}}}\right)\frac{1}{\sqrt{\omega_{\boldsymbol{k}}}}\Bigg|_{\eta=\eta_0} . \tag{5.24}$$

The first term agrees with the instantaneous-vacuum condition, while the second term is suppressed when the frequency changes adiabatically.

We can now approximate the vacuum energy per Fourier mode

$$E_{\boldsymbol{k}} \equiv \frac{1}{4}\left(|V_{\boldsymbol{k}}'|^2 + \omega_{\boldsymbol{k}}^2 |V_{\boldsymbol{k}}|^2\right) = \frac{1}{2}\omega_{\boldsymbol{k}} + \frac{1}{16}\frac{\omega_{\boldsymbol{k}}'^2}{\omega_{\boldsymbol{k}}^3} \approx \frac{1}{2}\omega_{\boldsymbol{k}} \ , \tag{5.25}$$

which is approximately the minimum energy per mode at $\eta = \eta_0$.

Similarly, we can define the second-order adiabatic vacuum by using now

$$\left(W_{\boldsymbol{k}}^{(2)}\right)^2 = \omega_{\boldsymbol{k}}^2 - \frac{1}{2}\left(\frac{\omega_{\boldsymbol{k}}''}{\omega_{\boldsymbol{k}}} - \frac{3}{2}\frac{\omega_{\boldsymbol{k}}'^2}{\omega_{\boldsymbol{k}}^2}\right) \ , \tag{5.26}$$

which on the other hand will differ from the instantaneous vacuum.

It is important to note that the adiabatic mode functions are approximate, but that they are only used to normalize the exact mode solution at $\eta = \eta_0$, and it is the exact mode solutions that are quantised. A'th order adiabatic vacuum is therefore mathematically a perfectly good consistent vacuum, not only in an approximate sense.

Standard Big Bang Model

In order to apply the methods of the previous sections to understand how particles and matter is produced in the early universe, let us first review some of the basic and most relevant aspects of cosmology. Assuming large-scale isotropy and homogeneity, we have, as earlier discussed, the FRW metric. This time we write it in spherical coordinates and allow for spatial curvature

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)\right] . \tag{6.1}$$

Inserting the metric into the left-hand side of the Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{6.2}$$

and the energy-momentum tensor for a comoving ideal fluid with energy density ρ and pressure p,

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \qquad T^\mu{}_\nu = \mathrm{diag}\,(-\rho, p, p, p)\,, \tag{6.3}$$

where $u^\mu = (1, 0, 0, 0)$ for a comoving observer, one obtains the Friedmann equations. From the time component ("00" component) of the Einstein equation, one obtains the first Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} \tag{6.4}$$

and from the spatial component ("ii" component) of the Einstein equation, one obtains the second Friedmann equation

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \ . \tag{6.5}$$

It is also useful to consider the comoving conservation of the energy-momentum tensor

$$D^\nu T_{\mu\nu} = 0 \tag{6.6}$$

which implies the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \ . \tag{6.7}$$

The continuity equation is not independent of the two Friedmann equations. Only two of the equations are independent, and the third can be derived from the others.

If we assume a specific equation of state, $p = w\rho$, with constant equation of state parameter, w, then from the continuity equation, we have

$$\rho \propto a^{-3(w+1)} \tag{6.8}$$

and then from the Friedmann equation, we have, in a spatially flat universe dominated by this component, for $w \neq -1$,

$$a \propto t^{\frac{2}{3(1+w)}} \ . \tag{6.9}$$

For $w = -1$, instead, $\rho = const.$ and $a(t)$ grows exponentially. So the continuity equation tells us how the energy density dilutes with the expansion of the universe, and the Friedmann equation gives us the scale factor as a function of time.

Some important examples are

$$\rho \propto a^{-3(\omega+1)} \Rightarrow \begin{cases} \text{Vacuum energy}: & \omega = -1, \quad \rho_\Lambda \sim const. \\ \text{Radiation}: & \omega = \frac{1}{3}, \quad \rho_r \sim a^{-4} \\ \text{Matter/dust}: & \omega = 0, \quad \rho_m \sim a^{-3} \end{cases} \tag{6.10}$$

We note that as we go back in time and the scale factor shrinks, the radiation component becomes more and more important, and we conclude that among the standard matter, radiation, and vacuum-energy components, radiation dominates sufficiently far back in the hot Big Bang era.

To parametrise the departure from a spatially flat universe, it is useful to define the critical energy density

$$\rho_c = \frac{3}{8\pi G_N} H^2 \tag{6.11}$$

which is the energy density we would infer from a measure of the expansion rate if the universe is assumed to be flat. We then define the ratio between the actual energy density and the critical energy density

$$\Omega = \frac{\rho}{\rho_c} \tag{6.12}$$

and the curvature is then related to the departure of this quantity from one

$$\Omega - 1 = \frac{k}{a^2 H^2} \ . \tag{6.13}$$

For $\Omega < 1$ we will have an open universe with a hyperbolic geometry. For $\Omega = 1$, we will have a flat universe, and for $\Omega > 1$, we will have a closed universe with the geometry of a hypersphere. This closed universe is finite in extent, like the area of an ordinary sphere. Just like a sphere is a two-dimensional surface embedded in 3 Euclidean spatial dimensions, the spatial section of a closed universe is the 3-dimensional surface of a hypersphere embedded in 4 Euclidean spatial dimensions.

6.1 Particle Horizon

For a radiation-dominated universe, the energy density blows up as the scale factor and time go to zero, a finite time ago. So in the standard Big Bang model, the universe started in a radiation-dominated universe after the Big Bang, a finite time ago. This also implies that photons have only travelled a finite distance since the Big Bang, and therefore the observable universe is finite even if the universe as a whole is infinite. To make this more precise, it is useful to introduce the particle horizon concept.

Photons travel along null geodesics with $ds^2 = 0$. For a radial null ray in the curved FRW metric, it is convenient to define the comoving radial coordinate χ by

$$d\chi \equiv \frac{dr}{\sqrt{1-kr^2}} \ , \tag{6.14}$$

so that the null condition gives

$$d\chi = \frac{dt}{a\,(t)} \ . \tag{6.15}$$

For a spatially flat universe, $k = 0$, we simply have $\chi = r$ and hence $dr = dt/a(t)$.

The physical radial distance on a constant-time hypersurface is $a(t)\chi$, and the physical particle-horizon distance at time t is therefore

$$d_H\,(t) = a\,(t) \int_{t_i}^{t} \frac{dt'}{a\,(t')} \ , \tag{6.16}$$

where t_i denotes the earliest time to which the classical cosmological description is extrapolated.

In the radiation-dominated era,

$$a\,(t) \propto t^{1/2}, \qquad H = \frac{1}{2t}, \tag{6.17}$$

and, taking $t_i = 0$ within the classical solution,

$$d_H\,(t) = a\,(t) \int_0^t \frac{dt'}{a\,(t')} = 2t = \frac{1}{H} \ . \tag{6.18}$$

More generally, if $a(t) \propto t^\beta$ with $0 < \beta < 1$,

$$d_H\,(t) = \frac{t}{1-\beta} = \frac{\beta}{1-\beta} H^{-1}. \tag{6.19}$$

Thus the particle horizon and the Hubble radius H^{-1} are distinct concepts, although they are of the same order of magnitude in the standard decelerating power-law eras.

If a physical scale λ is much larger than the particle horizon, it is out of causal contact, as no information can have propagated on those distance scales since the beginning of the classical cosmological evolution.

6.2 Horizon Problem

At recombination, around 380000 years after the Big Bang, the universe had cooled sufficiently for electrons and nuclei, primarily protons, to form electrically neutral atoms. Before this time, photons had enough energy to ionise the hydrogen atom into free electrons and protons, and as the temperature fell, protons combined with electrons into electrically neutral hydrogen atoms. At this point, the density of free electrons dropped dramatically, so photons ceased to scatter efficiently through Thomson scattering and decoupled from the baryonic matter, making the universe transparent to photons, and leaving us with the Cosmic Microwave Background (CMB) radiation as a snapshot of the universe only 380000 years after the Big Bang.

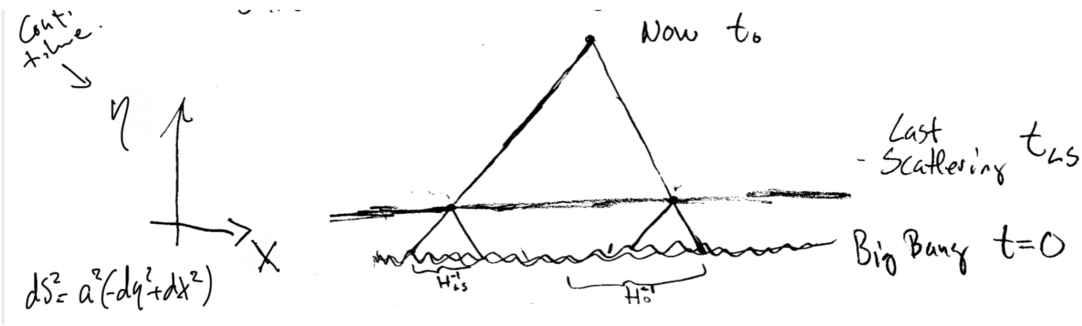


Figure 2: Illustration for the horizon problem.

The CMB shows a blackbody spectrum with a temperature around $2.725\,K$, homogeneous to within roughly one part in 10^5 after removing the much larger dipole anisotropy. The extreme homogeneity of the CMB actually poses a problem: the horizon problem. One can see the problem by comparing the particle horizon today with the particle horizon at last scattering.

The physical distance travelled by a photon since the Big Bang is called the particle horizon. Using $d\chi = dt/a(t)$ for photons, we have

$$d_H(t) = a(t) \cdot \int_0^t \frac{dt'}{a\left(t'\right)} = \frac{\beta}{1-\beta} H^{-1}, \quad a(t) \propto t^{\beta}\,, \quad 0 < \beta < 1\ . \tag{6.20}$$

Last scattering was at $z \simeq 1100$. The physical size at last scattering of the region that evolves into our present observable region is approximately

$$\lambda_H\left(t_{LS}\right) = d_H\left(t_0\right)\left(\frac{a_{LS}}{a_0}\right) = d_H\left(t_0\right)\left(\frac{T_0}{T_{LS}}\right), \tag{6.21}$$

where we have used $T \propto a^{-1}$ for the photon temperature.

We can assume approximately that the universe was matter-dominated at last scattering

$$H^2 \propto \rho_m \propto a^{-3} \propto T^3, \tag{6.22}$$

which implies that the Hubble radius scales as

$$H_{LS}^{-1} \propto \left(\frac{T_{LS}}{T_0}\right)^{-3/2}. \tag{6.23}$$

The particle horizon during matter domination is $d_H = 2H^{-1}$, so for an order-of-magnitude estimate it has the same scaling. If we make the crude approximation that this scaling may be used to compare the last-scattering and present particle horizons,

$$d_H\left(t_{LS}\right) \sim d_H\left(t_0\right)\left(\frac{T_{LS}}{T_0}\right)^{-3/2}. \tag{6.24}$$

This implies that the length corresponding to our present observable region was much larger than the particle-horizon size at last scattering. Comparing volumes of the two scales,

$$\frac{\lambda_H^3\left(t_{LS}\right)}{d_H^3\left(t_{LS}\right)} = \left(\frac{T_0}{T_{LS}}\right)^{-3/2} \sim 4 \cdot 10^4\,, \tag{6.25}$$

using $T_{LS}/T_0 \simeq 1100$. A more accurate treatment including the radiation-to-matter transition and the late-time cosmological evolution changes the numerical factor but not the qualitative conclusion: we find that there were a large number of causally disconnected regions within the volume corresponding to our observable universe at the time of last scattering.

We observed that at last scattering the temperature at two points is the same, even if they can never have "talked" to each other (see figure above). This is the Horizon Problem. We should see no correlation of density fluctuations between seemingly completely uncorrelated universes.

6.3 A Solution to the Problem

The core of the problem is the finite particle horizon

$$d_H\left(t\right) = a\left(t\right)\int_0^t \frac{dt'}{a\left(t'\right)} = a\left(t\right)\int_0^a \frac{da'}{a\dot{a}}\,. \tag{6.26}$$

We notice that in a radiation-dominated universe undergoing decelerated expansion with $\ddot{a} < 0$, $\dot{a}$ is larger in the past, and the integral converges in the lower limit, while if we would have accelerated expansion with $\ddot{a} > 0$, then $\dot{a}$ is smaller in the past, and the integral diverges in the lower limit, solving the problem.

Therefore, an early phase just after the Big Bang, before radiation domination, with $\ddot{a} > 0$, will push out the particle horizon, solving the horizon problem. This is also known as cosmic inflation.

From the second Friedmann equation

$$\frac{\ddot{a}}{a} = \frac{-4\pi G_N}{3}(\rho + 3p) \tag{6.27}$$

we have $\ddot{a} > 0$ implies $\rho + 3p < 0$, or, for $\rho > 0$,

$$p < -\frac{\rho}{3}. \tag{6.28}$$

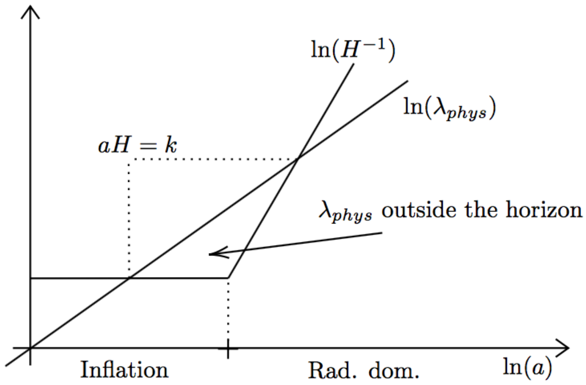


Figure 3: Illustration for the particle horizon.

In realistic models of inflation, we have an approximate vacuum-energy-type equation of state, $p \simeq -\rho$. The continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{6.29}$$

then implies $\rho \approx const.$. Then, from the first Friedmann equation for negligible spatial curvature during inflation

$$H^2 = \frac{8\pi G_N}{3}\rho \tag{6.30}$$

we have $H \approx const.$ during inflation. In the figure above, it is illustrated how this solves the horizon problem by placing a super-horizon scale at last scattering inside the horizon initially during inflation, such that initial conditions can be set on sub-horizon scales at the beginning of inflation and explain why all of the universe is so remarkably correlated and homogeneous at the time of last scattering.

Finally, let us note that inflation with nearly constant H corresponds to an approximately exponential expansion

$$\frac{\dot{a}}{a} \simeq const. \quad \Rightarrow \quad a \simeq a_i e^{H_I(t-t_i)} \ . \tag{6.31}$$

Exact constant H gives de Sitter space-time in the spatially flat slicing, while realistic slow-roll inflation is approximately de Sitter (quasi-de Sitter).

The Inflaton

To have a microphysical understanding of inflation, we need a field theory description of a fluid with $p < -\rho/3$. The simplest possibility is to consider a scalar field ϕ, called the inflaton. The minimally coupled inflaton has the action

$$\begin{aligned} S_\phi &= \int d^4\mathbf{x}\sqrt{-g}\mathcal{L} \\ &= -\int d^4\mathbf{x}\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] \\ &= \int dt d^3\boldsymbol{x} a^3\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial_i\phi\partial^i\phi - V(\phi)\right] \end{aligned} \tag{7.1}$$

Using Euler-Lagrange, we get the equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V_\phi(\phi) = 0 \tag{7.2}$$

where $V_\phi = \frac{dV}{d\phi}$. The energy-momentum tensor of the scalar field is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \ . \tag{7.3}$$

Now using[3]

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \ . \tag{7.8}$$

[3]Jacobi's formula for the variation of a determinant gives

$$\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}. \tag{7.4}$$

Hence

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}. \tag{7.5}$$

Since $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$, variation also gives

$$\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}, \tag{7.6}$$

and therefore, when varying with respect to the inverse metric,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{7.7}$$

This is the form needed directly in the definition of $T_{\mu\nu}$ above.

Applying this relation together with $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ when varying with respect to $g^{\mu\nu}$ then yields

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L} \; . \tag{7.9}$$

If we compare with the energy-momentum tensor of an ideal fluid $T_{\mu\nu} = \text{diag}\left(\rho, a^2p, a^2p, a^2p\right)$, we can make the following identifications

$$\begin{aligned} T_{00} = \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V\left(\phi\right) + \frac{1}{2}\frac{1}{a^2}\left(\nabla\phi\right)^2 \\ p_\phi \equiv \frac{1}{3}T^i{}_i &= \frac{1}{2}\dot{\phi}^2 - V\left(\phi\right) - \frac{1}{6}\frac{1}{a^2}\left(\nabla\phi\right)^2 \; . \end{aligned} \tag{7.10}$$

For a generic inhomogeneous field, T_{ij} need not have the perfect-fluid form point by point. The expression above is the isotropic spatial pressure obtained from one third of the spatial trace.

Notice if the gradient term dominated, we would have $p_\phi = -\frac{1}{3}\rho_\phi$, which is not quite enough to drive inflation. Also, if the kinetic term dominates, we get a stiff fluid with $p_\phi = \rho_\phi$, which obviously is not good for describing inflation, but if the potential term dominates, we have an equation of state $p_\phi = -\rho_\phi$ as desired. We are therefore interested in describing inflation by a homogeneous potential-energy-dominated scalar field. Let us therefore expand the quantum field as a perturbation around a homogeneous classical background

$$\hat{\phi}\left(\boldsymbol{x}, t\right) = \phi_c\left(t\right) + \delta\hat{\phi}\left(\boldsymbol{x}, t\right) \tag{7.11}$$

with the background defined by

$$\langle 0|\hat{\phi}\left(t, \boldsymbol{x}\right)|0\rangle = \phi_c\left(t\right) , \qquad \langle 0|\delta\hat{\phi}\left(\boldsymbol{x}, t\right)|0\rangle = 0 \; . \tag{7.12}$$

More generally, the expectation values here may be understood as being taken in the chosen quantum state; it need not be the exact interacting vacuum.

The classical background describes space-time well when quantum fluctuations are small, and the energy-momentum tensor for the classical background becomes

$$\begin{aligned} T_{00} = \rho_{\phi_c} &= \frac{1}{2}\dot{\phi}_c^2 + V\left(\phi_c\right) \\ \frac{T_{ii}}{a^2} = p_{\phi_c} &= \frac{1}{2}\dot{\phi}_c^2 - V\left(\phi_c\right) \end{aligned} \tag{7.13}$$

For the case with $V(\phi_c) \gg \dot{\phi}_c^2$ we now have $p_{\phi_c} \simeq -\rho_{\phi_c}$, which gives us inflation. In fact, for a homogeneous scalar field

$$\frac{\ddot{a}}{a} = \frac{8\pi G_N}{3}\left(V - \dot{\phi}_c^2\right) , \tag{7.14}$$

so accelerated expansion requires $V > \dot{\phi}_c^2$, while slow-roll inflation assumes the stronger condition $\dot{\phi}_c^2 \ll V$. Below, we will study the necessary conditions for inflation in more detail.

7.1 Slow-roll Conditions

Let us study the equation of motion for the background

$$\ddot{\phi}_c + 3H\dot{\phi}_c + V'(\phi_c) = 0 \ . \tag{7.15}$$

If we assume a slow roll with $\dot{\phi}^2 \ll V(\phi)$, then, unless we also assume $\ddot{\phi}$ to be negligible, the derivative will grow fast and the phase of inflation will be short. Thus to have a prolonged era of inflation, we will also assume $\left|\ddot{\phi}\right| \ll 3H\left|\dot{\phi}\right|$. This leads us to the slow-roll equation of motion

$$3H\dot{\phi} \simeq -V_\phi(\phi) \tag{7.16}$$

and

$$H^2 = \frac{8\pi G_N}{3}\rho_{\phi_c} \simeq \frac{8\pi G_N}{3}V(\phi_c) \tag{7.17}$$

when the inflaton dominates the total energy density.

The slow-roll equation of motion and the approximation of the energy density in terms of the potential energy are then valid when the following slow-roll conditions are satisfied

$$\begin{aligned} \dot{\phi}_c^2 \ll V(\phi) &\Rightarrow \frac{V_\phi^2}{V} \ll H^2 \\ \ddot{\phi}_c \ll 3H\dot{\phi}_c &\Rightarrow V_{\phi\phi} \ll H^2 \end{aligned} \tag{7.18}$$

where we used $d/d\phi = (1/\dot{\phi})d/dt$, when applying the chain rule. Rather than using these schematic relations, it is conventional to express these conditions through the dimensionless potential slow-roll parameters

$$\begin{aligned} \epsilon_V &\equiv \frac{1}{16\pi G_N}\left(\frac{V_\phi}{V}\right)^2 = \frac{M_{Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2, \\ \eta_V &\equiv \frac{1}{8\pi G_N}\frac{V_{\phi\phi}}{V} = M_{Pl}^2\frac{V_{\phi\phi}}{V}. \end{aligned} \tag{7.19}$$

For comparison, the Hubble slow-roll parameter is

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} = 4\pi G_N\frac{\dot{\phi}^2}{H^2}, \tag{7.20}$$

where the final equality is exact for a spatially flat universe dominated by a canonical homogeneous scalar field. Under the slow-roll approximation,

$$\epsilon_H \simeq \epsilon_V . \tag{7.21}$$

The relation

$$\eta_V \simeq \frac{1}{3}\frac{V_{\phi\phi}}{H^2} \tag{7.22}$$

also follows from $3M_{Pl}^2 H^2 \simeq V$.

In terms of these parameters, the slow roll conditions become

$$\epsilon_V \ll 1 \tag{7.23}$$

$$|\eta_V| \ll 1 \ . \tag{7.24}$$

Note, however, that formally we only require $\epsilon_H < 1$ to have inflation, since $\ddot{a} > 0$ is equivalent to $\epsilon_H < 1$. It is not called slow-roll inflation unless the slow-roll conditions are satisfied.

In order to understand how much inflation is needed, one can use that the scale factor has an approximately exponential growth when H varies slowly. Then in terms of e-foldings

$$N = \ln\left(\frac{a_f}{a_i}\right) = \int_{a_i}^{a_f} \frac{da}{a} = \int_{t_i}^{t_f} H dt \ . \tag{7.25}$$

Using the slow-roll equations, this can also be written as

$$N \simeq \frac{1}{M_{Pl}^2}\int_{\phi_f}^{\phi_i} \frac{V}{V_\phi} d\phi \ . \tag{7.26}$$

For the cosmological scales observed in the CMB, one typically requires of order 50–60 e-folds between their horizon exit and the end of inflation, although the precise number depends on the inflationary energy scale and the post-inflationary reheating history. A sufficiently long total period of inflation solves the horizon and flatness problems.

7.2 Inflationary Models

There are many types of inflation, since the number of potentials that can be constructed to satisfy the slow-roll conditions is infinite. It is, however, useful to consider two broad classes of inflation. Large field models and small field models.

The large-field models have a potential of the form

$$V(\phi) = \Lambda^4 \left(\frac{\phi}{\mu}\right)^p, \qquad p > 0 \tag{7.27}$$

such that the second slow-roll parameter becomes

$$\eta_\phi = M_{Pl}^2 \frac{V_{\phi\phi}}{V} = p(p-1)\frac{M_{Pl}^2}{\phi^2} \tag{7.28}$$

The slow roll condition $|\eta_\phi| \ll 1$ then leads to $\phi > M_{Pl}$ (for $p \neq 1$).

Small field models can be thought of as having a inverted potential of the form

$$V(\phi) = \lambda^4 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right], \qquad p > 0 \tag{7.29}$$

$$\eta_\phi = M_{Pl}^2 \frac{V_{\phi\phi}}{V} = -p(p-1)\left(\frac{\phi}{\mu}\right)^p \frac{M_{Pl}^2}{\phi^2} \tag{7.30}$$

For $\mu \ll M_{Pl}$ the slow roll condition $|\eta_\phi| \ll 1$ can be rewritten as

$$\phi \ll \left(\frac{\mu^p}{M_{Pl}^2}\right)^{\frac{1}{p-2}} = \left(\frac{\mu}{M_{Pl}}\right)^{\frac{2}{p-2}} \mu \ll \mu \tag{7.31}$$

for $p > 2$.

There are good and bad sides to both types of models, but the problems and the benefits are in some sense complementary. Large-field inflation is historically associated with the chaotic-inflation idea, in which inflation need not begin near a specially chosen local maximum of the potential but can start in regions where the scalar field initially takes sufficiently large values. At sufficiently high field values, some models can additionally enter a regime of *eternal inflation*, where quantum fluctuations of the field over a Hubble time become larger than its classical slow-roll displacement.

A separate issue in large-field models is the control of the effective field theory when the field excursion is comparable to or larger than M_{Pl}. A super-Planckian field value does not by itself invalidate effective field theory, but generic higher-dimensional operators can spoil the flatness of the potential unless their coefficients are sufficiently suppressed. Approximate shift symmetries or related structural mechanisms can protect the inflaton potential.

Small-field models, by contrast, often describe inflation near a local maximum or other special region of a potential and can arise in particle-physics motivated constructions. They typically involve an additional scale and may require more special initial conditions. The precise advantages and difficulties, however, depend on the detailed model rather than solely on whether the field range is classified as "large" or "small".

Cosmological Perturbations

The initial selling point of inflation is that it makes the universe nearly uniform. However, small quantum fluctuations of the inflaton are stretched to cosmological scales by the rapid expansion. This will induce ripples in spacetime due to the coupling of the inflaton fluctuations to the metric fluctuations in the Einstein equation. Since gravity couples to all matter, later on, the ripples will seed perturbations in photons (CMB), dark matter, and baryons. This is the blueprint for structure formation since dark matter and baryon over-densities become gravitationally unstable in the late universe and form the structures we observe, such as galaxies, galaxy clusters, and in the end planetary systems like our own.

In order to study how this mechanism for the creation of large-scale inhomogeneities in the universe works and study the spectrum and evolution of perturbations, we will first quantise a scalar field during inflation in this lecture. Then we will, in the next lecture, go on and take into account also the metric perturbations. Finally, we will briefly discuss the observational imprint on the CMB.

8.1 Quantum Fluctuations of a Scalar Field in Quasi-de Sitter Space

Let us consider a scalar field $\phi(\boldsymbol{x}, t)$ on a time-dependent background. We will as usual, split the field into a homogeneous background value and a small perturbation

$$\phi\left(\boldsymbol{x}, t\right) \equiv \phi_c\left(t\right) + \delta\phi\left(\boldsymbol{x}, t\right) \tag{8.1}$$

and then quantize $\delta\phi\left(\boldsymbol{x}, t\right)$. To proceed, we expand in Fourier modes

$$\delta\phi\left(\boldsymbol{x}, t\right) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \delta\phi_k\left(t\right) \ . \tag{8.2}$$

If ϕ is sub-dominant, meaning that the energy density of the field can be neglected compared to the total energy density in the universe and therefore has a negligible back-reaction on the metric, we can neglect metric perturbations. Perturbing the equation of motion for the field to linear order in perturbations,

$$\ddot{\phi}_c + \delta\ddot{\phi} + 3H\left(\dot{\phi}_c + \delta\dot{\phi}\right) - \frac{\nabla^2}{a^2}\delta\phi + V_\phi + V_{\phi\phi}\delta\phi = 0 \tag{8.3}$$

and applying the background equation of motion

$$\ddot{\phi}_c + 3H\dot{\phi}_c + V_\phi = 0 \ , \tag{8.4}$$

we get the linear equation of motion for the perturbations

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k + m^2\delta\phi_k = 0 \tag{8.5}$$

where the mass is, in general, an effective mass determined from the potential $m^2 \equiv V_{\phi\phi}$. This equation is only valid if the energy density of ϕ is negligible. Otherwise, it creates metric perturbations, which need to be included at the linear level as well.

It turns out to be convenient to switch to conformal time

$$\eta = \int \frac{1}{a}dt = \int \frac{1}{a\dot{a}}da = \int \frac{1}{a^2 H}da \tag{8.6}$$

If we consider pure de Sitter, then taking the Hubble parameter to be constant $H = const.$, we have

$$\eta = \frac{1}{H}\int \frac{1}{a^2}da = \frac{-1}{aH}\ , \tag{8.7}$$

where the integration constant has been chosen so that $\eta \to 0^-$ as $a \to \infty$.

For quasi-de Sitter expansion it is useful to introduce

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{8.8}$$

Using

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{1-\epsilon}{a}, \tag{8.9}$$

we find, for approximately constant ϵ,

$$\eta = -\frac{1}{aH\,(1-\epsilon)}\ . \tag{8.10}$$

When ϵ changes slowly, this relation remains valid to leading order in the slow-roll expansion.

If we also introduce the canonically normalized field, $\delta\chi = a\delta\phi$, then we have

$$\delta\chi_k'' + \left(k^2 + m^2a^2 - \frac{a''}{a}\right)\delta\chi_k = 0 \tag{8.11}$$

where " $'$ " marks derivatives with respect to conformal time. This equation has the familiar form of a harmonic oscillator with a time-dependent mass

$$M^2(\eta) \equiv m^2 a^2 - \frac{a''}{a} \qquad \left[m^2 = V_{\phi\phi}\right] \ . \tag{8.12}$$

Using that $\frac{\ddot{a}}{a} = \dot{H} + H^2$ and $\epsilon = -\frac{\dot{H}}{H^2}$ we get

$$\begin{aligned} \frac{a''}{a} &= a^2\left(\frac{\ddot{a}}{a} + H^2\right) = a^2\left(\dot{H} + 2H^2\right) \\ &= a^2(2-\epsilon)H^2 = \frac{(2-\epsilon)}{\eta^2(1-\epsilon)^2} \simeq \frac{1}{\eta^2}(2+3\epsilon) \end{aligned} \tag{8.13}$$

Using this with (8.11) gives

$$\delta\chi_k'' + \left(k^2 + M^2(\eta)\right)\delta\chi_k = 0 \tag{8.14}$$

where

$$M^2(\eta) = \frac{1}{\eta^2}\left(\frac{m^2}{H^2} - 2 - 3\epsilon\right) + \mathcal{O}\left(\epsilon^2\right) \ . \tag{8.15}$$

Here we have also assumed that m^2/H^2 is of slow-roll order, so products such as $\epsilon m^2/H^2$ are of higher order.

We can bring this differential equation into a recognisable form if we define ν_ϕ by

$$\begin{aligned} M^2(\eta) &\equiv -\frac{1}{\eta^2}\left(\nu_\phi^2 - \frac{1}{4}\right) \\ \Rightarrow \quad \nu_\phi^2 &= \frac{1}{4} - \eta^2 M^2(\eta) = \frac{9}{4} + 3\epsilon - \frac{m^2}{H^2} \ . \end{aligned} \tag{8.16}$$

The differential equation now becomes a Bessel equation in reduced form

$$\delta\chi_k'' + \left[k^2 - \frac{1}{\eta^2}\left(\nu_\phi^2 - \frac{1}{4}\right)\right]\delta\chi_k = 0 \ , \tag{8.17}$$

which has a generic solution in terms of Hankel functions for real ν_ϕ

$$\delta\chi_k = \sqrt{-\eta}\left[A_k H^{(1)}_{\nu_\phi}(-k\eta) + B_k H^{(2)}_{\nu_\phi}(-k\eta)\right] \tag{8.18}$$

and the Hankel functions $H^{(1)}_\nu$ and $H^{(2)}_\nu$ are defined in terms of Bessel functions of the first and second kind as

$$H^{(1)}_\nu(x) = J_\nu(x) + iY_\nu(x) \, , \quad H^{(2)}_\nu(x) = J_\nu(x) - iY_\nu(x) \tag{8.19}$$

Now we can quantise the fluctuation field in the standard way by promoting it to an operator $\hat{\delta\chi}_k$, which we expand in the mode functions

$$\delta\hat{\chi}_{\boldsymbol{k}} = \delta\chi_k \hat{a}^-_{\boldsymbol{k}} + \delta\chi^*_k \hat{a}^+_{-\boldsymbol{k}} \tag{8.20}$$

We impose[4]

$$\left[\hat{a}^-_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}'}\right] = \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right), \tag{8.21}$$

and canonical quantisation requires the Wronskian condition

$$\delta\chi_k \delta\chi^{*\prime}_k - \delta\chi^*_k \delta\chi'_k = i \ . \tag{8.22}$$

Thus the short-distance positive-frequency mode is normalized as $e^{-ik\eta}/\sqrt{2k}$.

8.2 Massless Scalar in Pure de Sitter

As a warm-up, let us inspect the solution in the case of pure de Sitter space. In that case we have $\nu_\phi^2 = \frac{9}{4}$ and the Hankel functions takes a particular simple form yielding

$$\delta\chi_k = \widetilde{A}_k \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + \widetilde{B}_k \left(1 + \frac{i}{k\eta}\right) e^{ik\eta} \tag{8.23}$$

On small sub-horizon scales $k \gg aH$ in the UV, we have $-\eta k \gg 1$ and thus

$$\delta\chi_k \approx \widetilde{A}_k e^{-ik\eta} + \widetilde{B}_k e^{ik\eta} \ . \tag{8.24}$$

If we take $\widetilde{A}_k = \frac{1}{\sqrt{2k}}$, $\widetilde{B}_k = 0$, then we recover the Minkowski vacuum in the infinite past

$$\delta\chi_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad \text{for } \eta \to -\infty \ . \tag{8.25}$$

This also corresponds to the zeroth-order adiabatic vacuum in the infinite past, where we recover the usual dispersion relation

$$\omega_k^2(\eta) = \boldsymbol{k}^2 + M^2(\eta) = \boldsymbol{k}^2 - \frac{2}{\eta^2} \to \boldsymbol{k}^2 \quad \text{for } \eta \to -\infty \ . \tag{8.26}$$

[4] In this lecture we use the standard normalisation in which there is no extra factor $1/\sqrt{2}$ outside the mode functions.

Notice the instantaneous vacuum only exists for $\omega_k^2 > 0$

$$\omega_k^2(\eta) = \boldsymbol{k}^2 - \frac{2}{\eta^2} > 0 \ , \tag{8.27}$$

so only for modes satisfying

$$-k\eta > \sqrt{2} \tag{8.28}$$

does the instantaneous harmonic-oscillator ground state exist. This is slightly stronger than the usual qualitative condition $k \gg aH$ for being deep inside the Hubble radius.

Remembering that in the present normalisation the instantaneous vacuum (up to a phase) is determined by

$$\delta\chi_k(\eta_0) = \frac{1}{\sqrt{2\omega_k(\eta_0)}}, \quad \frac{\delta\chi_k'(\eta_0)}{\delta\chi_k(\eta_0)} = -i\omega_k(\eta_0) \ , \tag{8.29}$$

and keeping in mind that $\eta_0 \to -\infty$, $\omega_k(\eta_0) \to k$, one can verify that the above normalisation indeed corresponds to the instantaneous vacuum at $\eta_0 = -\infty$, which is also called the Bunch–Davies vacuum,

$$\delta\chi_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \ , \qquad \text{for} \qquad \eta \to -\infty. \tag{8.30}$$

More precisely, the full Bunch–Davies mode is

$$\delta\chi_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{8.31}$$

On super-Hubble scales $-k\eta \ll 1$, we then find

$$\begin{aligned} \delta\chi_k &\simeq \frac{1}{\sqrt{2k}}\frac{-i}{k\eta} = \frac{i}{\sqrt{2k}}\frac{aH}{k} \\ \Rightarrow |\delta\phi_k| &\simeq \frac{H}{\sqrt{2k^3}} \ . \end{aligned} \tag{8.32}$$

At a finite time η_0 the instantaneous vacuum only exists for modes where $-k\eta_0 > \sqrt{2}$.
However, for $\eta_0 \to -\infty$ all fixed comoving modes can be taken in the instantaneous minimum energy configuration.

8.3 Generic Scalar in Quasi de Sitter

Now, let us consider the more general solution valid for slow-roll inflation

$$\delta\chi_k = \sqrt{-\eta}\left[A_k H^{(1)}_{\nu_\phi}(-k\eta) + B_k H^{(2)}_{\nu_\phi}(-k\eta)\right] . \tag{8.33}$$

Again, we can inspect the sub-horizon limit by using the large-argument limit of the Hankel functions

$$H^{(1)}_{\nu_\phi}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}e^{i\left(x-\frac{\pi}{2}\nu_\phi-\frac{\pi}{4}\right)}, \qquad H^{(2)}_{\nu_\phi}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}e^{-i\left(x-\frac{\pi}{2}\nu_\phi-\frac{\pi}{4}\right)} , \tag{8.34}$$

and normalize to the Bunch–Davies vacuum by requiring

$$\begin{aligned} \delta\chi_k &= \frac{1}{\sqrt{2k}}e^{-ik\eta}, \quad \text{for } -k\eta \gg 1 \\ \Rightarrow \quad A_k &= \frac{\sqrt{\pi}}{2}e^{i\left(\nu_\phi+\frac{1}{2}\right)\frac{\pi}{2}}, \quad B_k = 0 . \end{aligned} \tag{8.35}$$

Then, using the small-argument limit for the Hankel functions

$$H^{(1)}_{\nu_\phi}(x \ll 1) \sim -i\frac{\Gamma(\nu_\phi)}{\pi}\left(\frac{2}{x}\right)^{\nu_\phi} = \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu_\phi-\frac{3}{2}}\frac{\Gamma(\nu_\phi)}{\Gamma\left(\frac{3}{2}\right)}x^{-\nu_\phi} \tag{8.36}$$

for $\mathrm{Re}\,\nu_\phi > 0$, one can obtain the super-Hubble value of the mode function

$$\delta\chi_k = e^{i\left(\nu_\phi-\frac{1}{2}\right)\frac{\pi}{2}}2^{\nu_\phi-\frac{3}{2}}\frac{\Gamma(\nu_\phi)}{\Gamma\left(\frac{3}{2}\right)}\frac{1}{\sqrt{2k}}(-k\eta)^{\frac{1}{2}-\nu_\phi} . \tag{8.37}$$

so the mode function for the physical field $\delta\hat{\phi}_k$ is

$$|\delta\phi_k| \simeq 2^{\nu_\phi-\frac{3}{2}}\frac{\Gamma(\nu_\phi)}{\Gamma\left(\frac{3}{2}\right)}\frac{H(1-\epsilon)}{\sqrt{2k^3}}(-k\eta)^{\frac{3}{2}-\nu_\phi} , \qquad -k\eta \ll 1 . \tag{8.38}$$

Using $-k\eta \simeq k/[aH(1-\epsilon)]$, and dropping slow-roll corrections to the overall normalisation, this is commonly written as

$$|\delta\phi_k| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\phi} , \qquad \frac{k}{aH} \ll 1 . \tag{8.39}$$

8.4 Power Spectrum

For a generic quantity $g(\boldsymbol{x}, t)$ with

$$\hat{g}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{g}_{\boldsymbol{k}}(t) \tag{8.40}$$

the power spectrum is defined as

$$\langle 0|\hat{g}^\dagger_{\boldsymbol{k}_1} \hat{g}_{\boldsymbol{k}_2}|0\rangle \equiv \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2) \frac{2\pi^2}{k_1^3} P_g(k_1) \ . \tag{8.41}$$

Statistical homogeneity gives the momentum-conserving delta function, while statistical isotropy implies that the power spectrum depends only on $k = |\boldsymbol{k}|$. With the Fourier convention used above,

$$\langle 0|\hat{g}^2(\boldsymbol{x}, t)|0\rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{2\pi^2}{k^3} P_g(k) = \int \frac{dk}{k} P_g(k) \ . \tag{8.42}$$

Thus the generic relation between the variance of the variable $g(\boldsymbol{x}, t)$ and its power spectrum is

$$\langle 0|\hat{g}^2(\boldsymbol{x}, t)|0\rangle = \int \frac{dk}{k} P_g(k) \ . \tag{8.43}$$

Now in the case of our scalar field, we use

$$\delta\hat{\phi}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}} \left[\delta\phi_k \hat{a}^-_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \delta\phi^*_k \hat{a}^+_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{8.44}$$

Using $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$ and $\left[\hat{a}^-_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}'}\right] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$, we obtain

$$\langle 0|\delta\hat{\phi}^2(\boldsymbol{x}, t)|0\rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} |\delta\phi_k|^2 = \int \frac{dk}{k} \frac{k^3}{2\pi^2} |\delta\phi_k|^2 \ . \tag{8.45}$$

So we have

$$\begin{aligned} \langle 0|\delta\hat{\phi}^2|0\rangle &= \int \frac{dk}{k} \frac{k^3}{2\pi^2} |\delta\phi_k|^2 = \int \frac{dk}{k} P_{\delta\phi}(k) \\ \Rightarrow P_{\delta\phi}(k) &= \frac{k^3}{2\pi^2} |\delta\phi_k|^2 \ . \end{aligned} \tag{8.46}$$

When we insert the solution to the mode equation normalised to the Bunch–Davies vacuum, we then obtain, to leading order in slow roll and neglecting the slowly varying normalisation factor involving $\Gamma(\nu_\phi)$,

$$P_{\delta\phi}(k) \simeq \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_\phi} . \tag{8.47}$$

The power spectrum, if measured, contains information about the mass of the scalar field and the quasi-de Sitter background through the spectral index defined as

$$n_{\delta\phi} - 1 \equiv \frac{d\ln\left(P_{\delta\phi}(k)\right)}{d\ln(k)} \simeq 3 - 2\nu_\phi . \tag{8.48}$$

Now from (8.16), for a light scalar with m^2/H^2 and ϵ both small,

$$\nu_\phi = \sqrt{\frac{9}{4} + 3\epsilon - \frac{m^2}{H^2}} \simeq \frac{3}{2} + \epsilon - \frac{m^2}{3H^2} \tag{8.49}$$

so that

$$n_{\delta\phi} - 1 \simeq -2\epsilon + \frac{2m^2}{3H^2} . \tag{8.50}$$

For a genuinely sub-dominant spectator field, this is the appropriate result, and there is no reason in general to identify $m^2/(3H^2)$ with the inflaton potential slow-roll parameter η_V. If ϕ is instead the inflaton and slow roll holds, then $m^2 = V_{\phi\phi}$ gives $m^2/(3H^2) \simeq \eta_V$. However, in that case, neglecting the metric perturbations is not self-consistent.

After the scalar metric perturbations are included and the gauge-invariant curvature perturbation is quantised, the observable scalar spectral index for canonical single-field slow-roll inflation becomes

$$n_s - 1 \simeq -6\epsilon_V + 2\eta_V . \tag{8.51}$$

We will derive this result in the next lecture.

Metric Perturbations

So far we have considered fluctuations of the scalar field:

$$\delta\phi \Rightarrow \delta T_{\mu\nu} \tag{9.1}$$

If the scalar field dominates the energy density (like for inflation), then the Einstein equation gives us the perturbations of the metric.

$$\begin{aligned} \delta T_{\mu\nu} \Rightarrow \quad & \left[\delta\mathcal{R}_{\mu\nu} - \frac{1}{2}\delta\left(g_{\mu\nu}\mathcal{R}\right)\right] = -8\pi G_N \delta T_{\mu\nu} \\ & \Rightarrow \quad \delta g_{\mu\nu}\ . \end{aligned} \tag{9.2}$$

Here the minus sign is consistent with the curvature convention used in the earlier lectures. With the opposite convention for the Riemann tensor, the Einstein equation would instead be written with $+8\pi G_N T_{\mu\nu}$. The physical Friedmann and perturbation equations are unchanged.

Thus, the scalar-field and metric perturbations are coupled through the linearised Einstein and Klein–Gordon equations,

$$\delta\phi \Leftrightarrow \delta g_{\mu\nu}\ . \tag{9.3}$$

9.1 Linear Perturbation Theory

Perturbation around the FRW metric $g^{(0)}_{\mu\nu}$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}\left(\boldsymbol{x}, t\right)\ . \tag{9.4}$$

Generally $\delta g_{\mu\nu}\left(\boldsymbol{x}, t\right)$ can be parametrized in terms of scalar modes (spin 0), vector modes (spin 1) and tensor modes (spin 2). Vector perturbations are associated with transverse velocity/vorticity degrees of freedom of matter, while tensor perturbations describe gravitational waves.

For the tensor modes, true degrees of freedom can exist even in vacuum. In vacuum, the only propagating degrees of freedom of the metric are gravitational waves, which are two polarisations of the spin-two tensor modes, as we will see in a moment.

Write:

$$g_{\mu\nu}\left(\boldsymbol{x}, \eta\right) = g^{(0)}_{\mu\nu}\left(\eta\right) + \delta g_{\mu\nu} = a^2\left[\eta_{\mu\nu} + h_{\mu\nu}\right]\ . \tag{9.5}$$

With the inverse of the metric:

$$g^{\mu\nu}(\boldsymbol{x},\eta) = a^{-2}\left[\eta^{\mu\nu} - h^{\mu\nu}\right] + \mathcal{O}\left(h^2\right) \tag{9.6}$$

where the Minkowski metric is used for lowering and raising the indices on $h_{\mu\nu}$:

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\alpha}h_{\rho\alpha} \ . \tag{9.7}$$

We can now parametrize the metric degrees of freedom in terms of the following scalar, vector, and tensor functions

$$h_{\mu\nu} \equiv \begin{pmatrix} -2\phi & B_i \\ B_i & -2\psi\delta_{ij} + E_{ij} \end{pmatrix} \tag{9.8}$$

where E_{ij} is traceless, and ψ parametrizes the "trace degree of freedom".

Separation in Scalar, Vector and Tensor Parts

The vectors can be decomposed into a true vector and a scalar part. Helmholtz decomposition for a general vector u_i gives:

$$u_i = \partial_i V + V_i \tag{9.9}$$

where $\partial_i V$ is curl free with

$$\partial_i\partial_j V - \partial_j\partial_i V = 0 \ , \tag{9.10}$$

and V_i is divergence-free with

$$\partial^i V_i = 0 \ . \tag{9.11}$$

Using this for our vector part of the perturbation metric, we get:

$$B_i = B_i^S + B_i^V, \qquad B_i^S = \partial_i B, \qquad \partial^i B_i^V = 0 \ . \tag{9.12}$$

The tensor part can be decomposed into a scalar, a vector and a tensor part like:

$$E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T \ . \tag{9.13}$$

The scalar part is traceless and has the form:

$$E_{ij}^S = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E \ . \tag{9.14}$$

The vector part has the form:

$$E_{ij}^V = -\frac{1}{2}\left(\partial_j E_i + \partial_i E_j\right) \ . \tag{9.15}$$

For it to be traceless E_i must be divergence-free with

$$\partial^i E_i = 0 \ , \tag{9.16}$$

and therefore it is a pure vector. Finally, the tensor part must be divergence-free and traceless:

$$\partial^i E_{ij}^T = 0, \qquad {E^T}^i{}_i = 0 \ . \tag{9.17}$$

Now we can see that for the scalar quantities ϕ, ψ, B, and E we have 4 components, for the vector quantities B_i^V and E_i we have 4 components, and for the tensor quantity E_{ij}^T we have 2 components. Giving us a total of 10 components, as expected for a symmetric four-dimensional metric.

These component counts should not yet be interpreted as physical propagating degrees of freedom. Four infinitesimal coordinate transformations remove four gauge degrees of freedom, and the gravitational constraint equations remove four additional non-propagating degrees of freedom. Pure General Relativity therefore has only the two propagating tensor polarisations. In the scalar sector of single-field inflation, the scalar field supplies one additional physical propagating degree of freedom, which can be represented by the gauge-invariant Mukhanov–Sasaki variable.

To linear order in perturbation theory, the scalar, vector, and tensor parts decouple from one another on a homogeneous and isotropic FRW background.

9.2 Gauge Transformation

Consider the change of coordinates from an infinitesimal gauge transformation of the coordinates:

$$\widetilde{x}^\mu = x^\mu + \delta x^\mu \ . \tag{9.18}$$

Decompose the vector δx^μ in the usual way:

$$\begin{aligned} \delta x^0 &= \xi^0\left(x^\mu\right) \\ \delta x^i &= \xi^i\left(x^\mu\right) + \partial^i \xi\left(x^\mu\right), \qquad \partial_i \xi^i = 0 \ . \end{aligned} \tag{9.19}$$

With ξ^i being divergence-free. This shows that we have two scalar gauge degrees of freedom, which we can choose freely, ξ^0 and ξ.

We can use these gauge degrees of freedom to eliminate two of the scalar degrees of freedom, as they are not independent, but related through a coordinate transformation. This gives us $4-2=2$ degrees of freedom.

We also have $\delta\phi$, but that is determined by the other two through the Einstein equation, and therefore not an additional independent degree of freedom. For a single scalar field there is ultimately only one propagating scalar degree of freedom.

Gauge Transformation of Scalar Quantities

Let's consider a scalar function of the form

$$q(\eta, \boldsymbol{x}) = q_0(\eta) + \delta q(\eta, \boldsymbol{x}) \tag{9.20}$$

and see how it transforms under the gauge transformation $\eta \to \widetilde{\eta} = \eta + \xi^0(\eta, \boldsymbol{x})$.

$$\delta\widetilde{q}(\eta, \boldsymbol{x}) = \delta q(\eta, \boldsymbol{x}) - q_0'(\eta)\,\xi^0 \ . \tag{9.21}$$

Now let's do the gauge transformation of ϕ. For this we have $ds^2 = d\widetilde{s}^2$, since the line element is invariant under a coordinate transformation. Thus, restricting to the terms required to obtain the transformation of the 00 component,

$$a^2\left(\widetilde{x}^0\right)\left(1+2\widetilde{\phi}\right)\left(d\widetilde{x}^0\right)^2 = a^2\left(x^0\right)(1+2\phi)\left(dx^0\right)^2 \ . \tag{9.22}$$

To linear order,

$$\begin{aligned} a^2\left(\widetilde{x}^0\right) &= a^2\left(x^0\right) + 2aa'\xi^0 \\ d\widetilde{x}^0 &= \left(1+\xi^{0\prime}\right)dx^0 + \partial_i\xi^0\,dx^i \\ \Rightarrow \quad 1+2\phi &= 1+2\widetilde{\phi}+2\frac{a'}{a}\xi^0+2\xi^{0\prime} \ . \end{aligned} \tag{9.23}$$

and we infer the transformation rule

$$\widetilde{\phi} = \phi - \xi^{0\prime} - \frac{a'}{a}\xi^0 \ . \tag{9.24}$$

A similar calculation gives the scalar gauge transformations

$$\begin{aligned}\widetilde{\phi} &= \phi - \xi^{0\prime} - \frac{a'}{a}\xi^0 \\ \widetilde{B} &= B + \xi^0 - \xi' \\ \widetilde{\psi} &= \psi + \frac{1}{3}\nabla^2\xi + \frac{a'}{a}\xi^0 \\ \widetilde{E} &= E - 2\xi \ .\end{aligned} \tag{9.25}$$

9.3 Gauge Invariant Quantities

We have seen that the interesting variables may assume different values in different gauges. This leaves an ambiguity if one discusses individual coordinate-dependent perturbations. It is useful to eliminate this ambiguity by identifying combinations that are gauge invariant. Then we can choose a gauge and calculate the gauge invariant quantity in that gauge.

Bardeen Potentials

One example of such gauge-invariant quantities is the Bardeen potentials. With the conventions above, they can be written

$$\begin{aligned}\Phi &= \phi + \frac{1}{a}\left[a\left(B - \frac{E'}{2}\right)\right]' \\ \Psi &= \psi + \frac{1}{6}\nabla^2 E - \frac{a'}{a}\left(B - \frac{E'}{2}\right) \ .\end{aligned} \tag{9.26}$$

The Bardeen potentials are gauge-invariant combinations of the scalar metric perturbations. In longitudinal gauge, $B = E = 0$, they reduce simply to $\Phi = \phi$ and $\Psi = \psi$.

Curvature Perturbations

The spatial curvature on hypersurfaces of constant conformal time η for a flat universe is, to linear order,

$${}^{(3)}\mathcal{R} = \frac{4}{a^2}\nabla^2\psi - \frac{2}{3a^2}\nabla^4 E \tag{9.27}$$

and in a gauge with $E = 0$ this reduces to ${}^{(3)}\mathcal{R} = 4\nabla^2\psi/a^2$. The curvature perturbation ψ itself is not gauge invariant.

Co-moving Curvature Perturbation

A convenient gauge-invariant definition of the curvature perturbation is

$$\mathcal{R} = \psi + \frac{a'}{a}\frac{\delta\phi}{\phi'} = \psi + H\frac{\delta\phi}{\dot{\phi}} \ . \tag{9.28}$$

It is gauge invariant, and if we make the gauge choice $\delta\phi = 0$ (comoving gauge),

$$\mathcal{R} = \psi|_{\delta\phi=0} \ , \tag{9.29}$$

we see that $\mathcal{R}$ can be understood as the curvature perturbation on a co-moving slice.

The curvature perturbation on spatial slices of uniform energy density is

$$\zeta = \psi + \frac{a'}{a}\frac{\delta\rho}{\rho'} = \psi + H\frac{\delta\rho}{\dot{\rho}} \ . \tag{9.30}$$

It is gauge invariant and

$$\zeta = \psi|_{\delta\rho=0} \ . \tag{9.31}$$

To see how to construct these gauge-invariant variables from the curvature perturbation in an arbitrary gauge, we find the gauge transformation that brings us back to the comoving gauge by choosing ξ^0 to cancel $\delta\phi$, and we have:

$$\psi \to \widetilde{\psi} = \psi + \frac{a'}{a}\delta\tau \ , \tag{9.32}$$

under $\eta \to \eta + \delta\tau$, where $\delta\tau \equiv \xi^0$ and $\xi = 0$.

Remember that a scalar transforms as

$$\delta\widetilde{q} = \delta q - \xi^0 q_0' \ . \tag{9.33}$$

Using this for $\delta\phi$ we get:

$$\delta\phi \to \delta\phi_{\mathrm{com}} = \delta\phi - \phi'\delta\tau = 0 \quad \Rightarrow \quad \delta\tau = \frac{\delta\phi}{\phi'} \ . \tag{9.34}$$

Likewise for $\delta\rho$ we get:

$$\delta\rho \to \delta\rho_{\mathrm{uniform}} = \delta\rho - \rho'\delta\tau = 0 \quad \Rightarrow \quad \delta\tau = \frac{\delta\rho}{\rho'} \ . \tag{9.35}$$

Which leads us to the transformation of ψ:

$$\psi \to \psi + \frac{a'}{a}\frac{\delta\phi}{\phi'} \qquad \vee \qquad \psi \to \psi + \frac{a'}{a}\frac{\delta\rho}{\rho'} \ . \tag{9.36}$$

9.4 Scalar Field Perturbation in a Spatial Flat Gauge

Similarly we can find the field perturbation in flat gauge $\psi_{\text{flat}} = 0$:

$$\psi \to \psi + \frac{a'}{a}\delta\tau = 0 \quad \Rightarrow \quad \delta\tau = -\frac{\psi}{\mathcal{H}} \ , \tag{9.37}$$

where

$$\mathcal{H} = \frac{a'}{a} = aH \tag{9.38}$$

is the conformal Hubble parameter. We can then write the field perturbation in flat gauge in a gauge invariant form

$$\delta\phi \to \delta\phi - \phi'\delta\tau = \delta\phi + \frac{\phi'}{\mathcal{H}}\psi \ . \tag{9.39}$$

This gauge-invariant field perturbation is often called the Sasaki–Mukhanov variable Q:

$$Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\psi = \delta\phi + \frac{\dot{\phi}}{H}\psi \equiv \frac{\dot{\phi}}{H}\mathcal{R} \ . \tag{9.40}$$

It is gauge invariant and

$$Q = \delta\phi|_{\psi=0} \ . \tag{9.41}$$

The canonically normalized Mukhanov–Sasaki variable is

$$v \equiv aQ = z\mathcal{R}, \qquad z \equiv a\frac{\dot{\phi}}{H} = \frac{a\phi'}{\mathcal{H}} \ . \tag{9.42}$$

9.5 Longitudinal Gauge

One often used gauge choice is to set $B = E = 0$. The two scalar metric functions remaining in this gauge are ϕ and ψ.

If the matter has no scalar anisotropic stress, then the traceless i, j Einstein equation implies

$$\psi = \phi \ . \tag{9.43}$$

A canonical scalar field has vanishing anisotropic stress at linear order, so this relation applies during single-field inflation.

Einstein Equation

The traceless i, j part $(i \neq j)$ gives:

$$\psi = \phi \ . \tag{9.44}$$

For the $0, i$ part we get:

$$\dot{\psi} + H\psi = 4\pi G_N \dot{\phi}\delta\phi = \epsilon H^2 \frac{\delta\phi}{\dot{\phi}} \ , \tag{9.45}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G_N \frac{\dot{\phi}^2}{H^2} \ . \tag{9.46}$$

For the $0, 0$ part we get:

$$-3H\left(\dot{\psi} + H\psi\right) + \frac{\nabla^2\psi}{a^2} = 4\pi G_N \left(\dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\psi + V'\delta\phi\right) \ . \tag{9.47}$$

where for a canonical scalar, $\delta\rho = \dot{\phi}\,\delta\dot{\phi} - \dot{\phi}^2\psi + V'\delta\phi$.

Finally, for the trace of the spatial i, i equation we have:

$$\ddot{\psi} + 4H\dot{\psi} + \left(2\dot{H} + 3H^2\right)\psi = 4\pi G_N \left(\dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\psi - V'\delta\phi\right) \ . \tag{9.48}$$

Using

$$\dot{H} = -4\pi G_N \dot{\phi}^2 \tag{9.49}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{9.50}$$

to eliminate the matter perturbations. One obtains

$$\ddot{\psi}_k + \left(H - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\psi}_k + 2\left(\dot{H} - H\frac{\ddot{\phi}}{\dot{\phi}}\right)\psi_k + \frac{k^2}{a^2}\psi_k = 0 \ . \tag{9.51}$$

In terms of slow-roll parameters (ϵ, η)[5] we use, to first order,

$$\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \epsilon - \eta \ , \tag{9.52}$$

and in conformal time the equation becomes

$$\psi_k'' + 2\mathcal{H}\left(\eta - \epsilon\right)\psi_k' + 2\mathcal{H}^2\left(\eta - 2\epsilon\right)\psi_k + k^2\psi_k = 0 \ . \tag{9.53}$$

[5]Here η denotes the potential slow-roll parameter, whereas elsewhere η also denotes conformal time. The meaning should be clear from context.

Super-horizon scales

On super-Hubble scales, $k \ll aH$, and formally taking the leading slow-roll limit $\epsilon, \eta \to 0$,

$$\psi_k'' \simeq 0 \quad \Leftrightarrow \quad a\frac{d}{dt}\left(a\dot{\psi}_k\right) = a^2\left(H\dot{\psi}_k + \ddot{\psi}_k\right) \simeq 0\ . \tag{9.54}$$

The non-decaying mode therefore satisfies

$$\dot{\psi}_k \simeq 0\ . \tag{9.55}$$

From this the $0, i$ part reduces to:

$$\psi_k \simeq \epsilon H\frac{\delta\phi_k}{\dot{\phi}} \tag{9.56}$$

on super-Hubble scales.

$$\mathcal{R}_k = \psi_k + H\frac{\delta\phi_k}{\dot{\phi}} = (1+\epsilon)\,H\frac{\delta\phi_k}{\dot{\phi}} \simeq H\frac{\delta\phi_k}{\dot{\phi}} \tag{9.57}$$

Using this, we can compute the power spectrum:

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}\left|\delta\phi_k\right|^2 = \frac{k^3}{4\pi^2}\frac{1}{M_{Pl}}\frac{1}{\epsilon}\left|\delta\phi_k\right|^2 \tag{9.58}$$

Perturbed Klein-Gordon:

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k + V''\delta\phi_k = -2\psi_k V' + 4\dot{\psi}_k\dot{\phi}_c \tag{9.59}$$

Use

$$\left|4\dot{\psi}_k\dot{\phi}_c\right| \ll \left|\psi_k V'\right|, \quad \psi_k = \epsilon H\frac{\delta\phi_k}{\dot{\phi}_c}, \quad V' \simeq -3H\dot{\phi}_c \tag{9.60}$$

With this, we get:

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(V'' + 6\epsilon H^2\right)\delta\phi_k = 0 \tag{9.61}$$

Define $\chi_k \equiv \delta\phi_k/a$:

$$\delta\chi_k'' - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\delta\chi_k = 0 \qquad \text{Where: } \nu^2 = \frac{9}{4} + 9\epsilon - 3\eta \tag{9.62}$$

For super-Hubble scales, we have:

$$|\delta\phi_k| \simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \tag{9.63}$$

With the power spectrum:

$$P_{\mathcal{R}}(k) = \frac{1}{2M_{Pl}^2\epsilon} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \tag{9.64}$$

$$n_{\mathcal{R}} - 1 = \frac{d\ln(P_{\mathcal{R}})}{d\ln(k)} = 3 - 2\nu = 2\eta - 6\epsilon \tag{9.65}$$

9.6 Co-moving Curvature Perturbation

Alternatively, we can use the gauge-invariant relation derived earlier

$$\mathcal{R} = \frac{H}{\dot{\phi}} Q \ . \tag{9.66}$$

and the curvature power spectrum may be written

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \frac{H^2}{\dot{\phi}^2} |Q_k|^2 = \frac{k^3}{4\pi^2 M_{Pl}^2 \epsilon} |Q_k|^2 \ . \tag{9.67}$$

To determine Q_k through horizon crossing, one quantises the canonical Mukhanov–Sasaki variable

$$v = aQ = z\mathcal{R}, \qquad z = a\frac{\dot{\phi}}{H} \ . \tag{9.68}$$

The quadratic action for scalar perturbations in the spatially flat gauge is

$$S^{(2)} = \frac{1}{2} \int d\eta \, d^3\boldsymbol{x} \left[v'^2 - (\nabla v)^2 + \frac{z''}{z} v^2 \right], \tag{9.69}$$

and the Fourier modes obey the Mukhanov–Sasaki equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \ . \tag{9.70}$$

To first order in the potential slow-roll parameters,

$$\frac{z''}{z} \simeq \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right), \qquad \nu \simeq \frac{3}{2} + 3\epsilon - \eta, \tag{9.71}$$

or equivalently

$$\nu^2 \simeq \frac{9}{4} + 9\epsilon - 3\eta \ . \tag{9.72}$$

The mode equation therefore becomes

$$v_k'' + \left[k^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right] v_k = 0 \ . \tag{9.73}$$

Choosing the Bunch–Davies vacuum in the sub-Hubble limit, $v_k \to e^{-ik\eta}/\sqrt{2k}$ for $-k\eta \gg 1$, gives on super-Hubble scales

$$|Q_k| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \tag{9.74}$$

to leading order in the slowly varying normalisation.

The curvature power spectrum is therefore

$$P_{\mathcal{R}}(k) \simeq \frac{1}{2M_{Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \ . \tag{9.75}$$

At horizon exit, $k = aH$, this becomes

$$P_{\mathcal{R}}(k) \simeq \left.\frac{H^2}{8\pi^2 M_{Pl}^2\epsilon}\right|_{k=aH} \ . \tag{9.76}$$

The spectral index is

$$n_{\mathcal{R}} - 1 \equiv \frac{d\ln(P_{\mathcal{R}})}{d\ln(k)} = 3 - 2\nu \simeq 2\eta - 6\epsilon \ . \tag{9.77}$$

Thus, identifying $n_{\mathcal{R}}$ with the usual scalar spectral index n_s,

$$n_s - 1 \simeq -6\epsilon_V + 2\eta_V \ , \tag{9.78}$$

where at this order we have written the potential slow-roll parameters explicitly.

9.7 The Cosmic Microwave Background

For adiabatic perturbations, and in the absence of significant non-adiabatic pressure perturbations, the curvature perturbations satisfy on super-Hubble scales

$$\dot{\mathcal{R}}_k \simeq \dot{\zeta}_k \simeq 0 \ . \tag{9.79}$$

This means that the co-moving curvature perturbation is conserved on super-Hubble scales. For single-field inflation, this allows us to evolve the primordial perturbation through epochs whose detailed microphysics may be complicated, provided the evolution remains adiabatic. In models with entropy or isocurvature perturbations, $\mathcal{R}$ need not be conserved.

For a constant equation of state and negligible anisotropic stress, the super-Hubble Bardeen potential is related to the conserved curvature perturbation by

$$\Phi = \frac{3\,(1+w)}{5+3w}\mathcal{R} \ . \tag{9.80}$$

Thus during the radiation-dominated phase of the universe,

$$\Phi = \frac{2}{3}\mathcal{R}, \qquad P_\Phi = \left(\frac{2}{3}\right)^2 P_{\mathcal{R}} \ , \tag{9.81}$$

while during the matter-dominated phase,

$$\Phi = \frac{3}{5}\mathcal{R}, \qquad P_\Phi = \left(\frac{3}{5}\right)^2 P_{\mathcal{R}} \ . \tag{9.82}$$

On angular scales sufficiently large that the perturbations were outside the Hubble radius at last scattering, the ordinary Sachs–Wolfe effect gives the dominant primary temperature anisotropy. The intrinsic photon-temperature perturbation and the gravitational redshift combine to give[6]

$$\frac{\delta T}{T} = \frac{1}{3}\Phi = \frac{1}{5}\mathcal{R} \tag{9.83}$$

during matter domination. On the very largest scales, late-time evolution of the gravitational potential also generates an integrated Sachs–Wolfe contribution.

[6]For more details see, for example, the GR and cosmology course FY812 or Riotto, `https://arxiv.org/abs/hep-ph/0210162`.

The Unruh Effect

So far, we have discussed particle production in an expanding Universe, where the time-dependent geometry can mix positive- and negative-frequency modes and thereby change the natural particle interpretation with time. In this section, as a warm-up for discussing Hawking radiation of black holes, we will discuss another example where two different observers have different notions of the vacuum state. So, instead of the space-time changing over time, we will look at two different observers in flat Minkowski space, one who is at rest in Minkowski space and another who is uniformly accelerated, and see that their natural notions of particles and vacuum are different.

10.1 Rindler Space

To relate the vacuum state of an accelerated observer in Minkowski space to the vacuum of one who is at rest, we are first going to find the "co-moving" coordinate system of an accelerated observer in Minkowski space, which is the coordinate system in which the accelerated observer is at rest at a fixed spatial coordinate. This metric, describing the rest frame of the accelerated observer in Minkowski space, is called Rindler space.

For simplicity, we can assume that the observer is accelerating in the x-direction, but remains at rest in the other directions. Consider a two-dimensional Minkowski space-time:

$$ds^2 = -dt^2 + dx^2 \ . \tag{10.1}$$

To describe the accelerated observer, make the coordinate transformation:

$$\begin{aligned} t &= a^{-1} e^{a\xi} \sinh(a\tau) \\ x &= a^{-1} e^{a\xi} \cosh(a\tau) \\ \Rightarrow \quad ds^2 &= e^{2a\xi}\left(-d\tau^2 + d\xi^2\right) \end{aligned} \tag{10.2}$$

These coordinates cover the right Rindler wedge $x > |t|$. The parameter $a > 0$ fixes the normalisation of the Rindler time coordinate, and equals the proper acceleration for an observer at $\xi = 0$.

To review uniformly accelerated motion, let us separately parameterise a trajectory by its proper time s. For the four-velocity, we have by definition:

$$u^\mu u_\mu = -1, \qquad u^\mu \equiv \frac{dx^\mu}{ds} \ , \tag{10.3}$$

and for the four-acceleration in Minkowski Cartesian coordinates we have:

$$\begin{aligned} a^\mu &\equiv \frac{du^\mu}{ds} = \frac{d^2 x^\mu}{ds^2} \\ a^\mu a_\mu &\equiv \alpha^2 \ . \end{aligned} \tag{10.4}$$

More generally, four-acceleration is defined covariantly by $a^\mu = u^\nu D_\nu u^\mu$; in Cartesian Minkowski coordinates this reduces to the ordinary derivative above.

In the observer's instantaneous rest frame, we have

$$u^\mu = (1, 0) \tag{10.5}$$

and differentiating $u^\mu u_\mu = -1$ gives

$$u_\mu a^\mu = 0 \ . \tag{10.6}$$

Thus, in the instantaneous rest frame

$$a^\mu = (0, \alpha) \ , \tag{10.7}$$

where α is the magnitude of the proper acceleration.

Consider

$$\begin{aligned} a^\mu a_\mu = \alpha^2 \quad \Rightarrow \quad -\left(\frac{du^0}{ds}\right)^2 + \left(\frac{du^1}{ds}\right)^2 &= \alpha^2 \\ -\left(u^0\right)^2 + \left(u^1\right)^2 &= -1 \ . \end{aligned} \tag{10.8}$$

Isolating:

$$u^0 = \sqrt{1 + (u^1)^2}, \quad \frac{du^1}{ds} = \alpha\sqrt{1 + (u^1)^2} \ , \tag{10.9}$$

assuming $u^0 > 0$ and $\frac{du^1}{ds} > 0$. Solving,

$$u^1(s) \equiv \frac{dx}{ds} = \sinh(\alpha s) \ , \quad u^0(s) \equiv \frac{dt}{ds} = \cosh(\alpha s) \ . \tag{10.10}$$

Integrating gives us

$$x(s) = \frac{1}{\alpha}\cosh(\alpha s) \ , \quad t(s) = \frac{1}{\alpha}\sinh(\alpha s) \ , \tag{10.11}$$

where we recognize the hyperbolic trajectories from Eq. (10.1).

For a trajectory of constant Rindler coordinate ξ, comparison with Eq. (10.1) gives

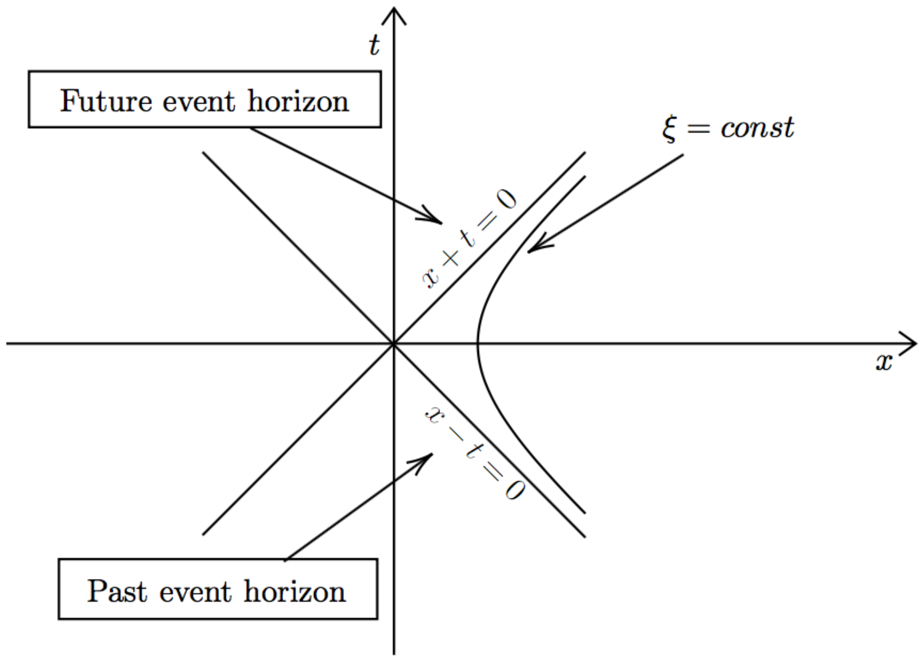
$$ds = e^{a\xi} d\tau, \qquad \alpha = a e^{-a\xi}, \qquad \alpha^{-1} = a^{-1} e^{a\xi} \ . \tag{10.12}$$

Thus τ is the proper time only for the observer at $\xi = 0$, whose proper acceleration is $\alpha = a$.

In Minkowski space, the accelerated observer at a constant ξ describes a hyperbola defined by the equation

$$x^2 - t^2 = a^{-2} e^{2a\xi} = \alpha^{-2} \ . \tag{10.13}$$

As illustrated in the figure, a worldline confined to the right Rindler wedge can never receive signals from part of Minkowski space-time or send signals to part of it. The null surfaces $x = \pm t$ form the past and future Rindler horizons for the uniformly accelerated observer.



10.2 Quantum Field in Rindler Space-Time

Now let us consider the vacuum state of a scalar field as seen by an accelerated observer versus an observer at rest in Minkowski space. The action of a massless scalar field in two space-time dimensions is:

$$S[\phi] = -\frac{1}{2}\int d^2x\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{10.14}$$

This is classically Weyl invariant in two dimensions under $g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$.

In the Minkowski coordinate system, we have

$$S[\phi] = \frac{1}{2}\int \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] dtdx \ , \tag{10.15}$$

while in the co-moving frame of the accelerated observer (Rindler coordinates) we have

$$S[\phi] = \frac{1}{2}\int \left[(\partial_\tau\phi)^2 - (\partial_\xi\phi)^2\right] d\tau d\xi \ . \tag{10.16}$$

Similarly, in the Minkowski frame, the equation of motion becomes

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \quad \Rightarrow \quad \bar{U}_k = \frac{1}{\sqrt{2|k|}}e^{ikx-i|k|t} \ , \tag{10.17}$$

while in the Rindler frame, the equation of motion is

$$\frac{\partial^2\phi}{\partial \tau^2} - \frac{\partial^2\phi}{\partial \xi^2} = 0 \quad \Rightarrow \quad U_k = \frac{1}{\sqrt{2|k|}}e^{ik\xi-i|k|\tau} \ , \tag{10.18}$$

where U_k and $\bar{U}_k$ are the usual positive frequency mode functions in the two frames, from which we obtain the two mode expansions. For the lab frame (Minkowski rest frame), we have:

$$\hat{\phi}(t,x) = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2|k|}}\left[e^{-i|k|t+ikx}\hat{a}_k^- + e^{i|k|t-ikx}\hat{a}_k^+\right] \ . \tag{10.19}$$

For the Rindler frame (rest frame of the accelerated observer) we have:

$$\hat{\phi}(\tau,\xi) = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2|k|}}\left[e^{-i|k|\tau+ik\xi}\hat{b}_k^- + e^{i|k|\tau-ik\xi}\hat{b}_k^+\right] \ . \tag{10.20}$$

This implies that we have two notions of a vacuum. The Minkowski vacuum and the right-Rindler vacuum are defined by

$$\hat{a}_k^-|0_M\rangle = 0, \qquad \hat{b}_k^-|0_R\rangle = 0 \ , \tag{10.21}$$

and they are not the same:

$$|0_M\rangle \neq |0_R\rangle \ . \tag{10.22}$$

Strictly speaking, a global Rindler description involves both the right and left Rindler wedges. The Minkowski vacuum is an entangled state of the left- and right-wedge Rindler degrees of freedom. When observables are restricted to one wedge, the Minkowski vacuum is described by a thermal density matrix. This is the origin of the Unruh effect.

10.3 Light-Cone Mode Expansion

In order to find the Bogoliubov transformation between the two particle descriptions, it is convenient to define the light-cone coordinates:

$$\begin{aligned} \bar{u} &\equiv t - x, \quad \bar{v} \equiv t + x \\ u &\equiv \tau - \xi, \quad v \equiv \tau + \xi \ . \end{aligned} \tag{10.23}$$

where the bar denotes the Minkowski coordinates, and without the bar, the Rindler coordinates. The coordinate transformation written in light cone coordinates becomes:

$$\begin{aligned} & \bar{u} = -\frac{1}{a}e^{-au}, \quad \bar{v} = \frac{1}{a}e^{av} \\ \Rightarrow \quad & ds^2 = -d\bar{u}\, d\bar{v} = -e^{a(v-u)}du\, dv \end{aligned} \tag{10.24}$$

For each coordinate set, we have an equation of motion and a corresponding solution.

$$\begin{aligned} \partial_{\bar{u}}\partial_{\bar{v}}\phi\left(\bar{u},\bar{v}\right) = 0, \qquad & \phi\left(\bar{u},\bar{v}\right) = A\left(\bar{u}\right) + B\left(\bar{v}\right) \\ \partial_u\partial_v\phi\left(u,v\right) = 0, \qquad & \phi\left(u,v\right) = P\left(u\right) + Q\left(v\right) \end{aligned} \tag{10.25}$$

We will now examine the functions $A(\bar{u})$, $B(\bar{v})$, $P(u)$ and $Q(v)$.

10.4 Mode Expansions

Splitting the Minkowski expansion into the $k > 0$ and $k < 0$ sectors, and writing $\omega = |k| > 0$, gives

$$\hat{\phi}\left(\bar{u},\bar{v}\right) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega\bar{u}}\hat{a}^-_\omega + e^{i\omega\bar{u}}\hat{a}^+_\omega + e^{-i\omega\bar{v}}\hat{a}^-_{-\omega} + e^{i\omega\bar{v}}\hat{a}^+_{-\omega}\right]. \tag{10.26}$$

Doing the same for the Rindler frame using $\Omega = |k| > 0$ we get:

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ + e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+ \right]. \tag{10.27}$$

Comparing with the solution in (10.25) we get in the Minkowski frame:

$$\begin{aligned}
\hat{\phi}(\bar{u},\bar{v}) &= \hat{A}(\bar{u}) + \hat{B}(\bar{v}) \\
\hat{A}(\bar{u}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ \right] \\
\hat{B}(\bar{v}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+ \right].
\end{aligned} \tag{10.28}$$

Similarly, in the Rindler frame, we get:

$$\begin{aligned}
\hat{\phi}(u,v) &= \hat{P}(u) + \hat{Q}(v) \\
\hat{P}(u) &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ \right] \\
\hat{Q}(v) &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+ \right].
\end{aligned} \tag{10.29}$$

Now write (coordinate transformation, do not mix u and v)

$$\begin{aligned}
&\hat{\phi}(u,v) = \hat{A}(\bar{u}(u)) + \hat{B}(\bar{v}(v)) = \hat{P}(u) + \hat{Q}(v) \\
&\Rightarrow \quad \hat{A}(\bar{u}(u)) = \hat{P}(u), \quad \hat{B}(\bar{v}(v)) = \hat{Q}(v) \ .
\end{aligned} \tag{10.30}$$

$$\begin{aligned}
\hat{A}(\bar{u}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ \right] \\
= \hat{P}(u) &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ \right].
\end{aligned} \tag{10.31}$$

Make a Fourier transformation in u on both sides. For the right-hand side we have:

$$\int_{-\infty}^\infty \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{P}(u) = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \hat{b}_\Omega^-, & \Omega > 0 \\ \hat{b}_{|\Omega|}^+, & \Omega < 0 \end{cases}. \tag{10.32}$$

For the left-hand side we have:

$$\begin{aligned}
\int_{-\infty}^\infty \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{A}(\bar{u}(u)) &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \int_{-\infty}^\infty \frac{du}{2\pi} \left[e^{i\Omega u - i\omega\bar{u}(u)} \hat{a}_\omega^- + e^{i\Omega u + i\omega\bar{u}(u)} \hat{a}_\omega^+ \right] \\
&\equiv \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega,\Omega) \hat{a}_\omega^- + F(-\omega,\Omega) \hat{a}_\omega^+ \right].
\end{aligned} \tag{10.33}$$

Where

$$F(\omega,\Omega) \equiv \int_{-\infty}^{\infty} \frac{du}{2\pi} \exp\left[i\Omega u - i\omega\bar{u}\right] = \int_{-\infty}^{\infty} \frac{du}{2\pi} \exp\left[i\Omega u + i\frac{\omega}{a}e^{-au}\right] . \tag{10.34}$$

Comparing the right-hand side and the left-hand side yields:

$$\begin{aligned} \hat{b}_\Omega^- &= \int_0^\infty d\omega \left[\alpha_{\omega\Omega}\hat{a}_\omega^- + \beta_{\omega\Omega}\hat{a}_\omega^+\right] \\ \alpha_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}}F(\omega,\Omega)\,, \quad \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}F(-\omega,\Omega) \ . \end{aligned} \tag{10.35}$$

It is similar for $\hat{b}_\Omega^+$ using $F^*(\omega,\Omega) = F(-\omega,-\Omega)$.

Now, for the lowering and raising operators, we have the commutators:

$$\left[\hat{a}_\omega^-, \hat{a}_{\omega'}^+\right] = \delta(\omega-\omega')\,, \quad \left[\hat{b}_\Omega^-, \hat{b}_{\Omega'}^+\right] = \delta(\Omega-\Omega') \tag{10.36}$$

Using this, we get:

$$\int_0^\infty d\omega \left(\alpha_{\omega\Omega}\alpha_{\omega\Omega'}^* - \beta_{\omega\Omega}\beta_{\omega\Omega'}^*\right) = \delta(\Omega-\Omega') \ . \tag{10.37}$$

Now, with the Rindler number operator in the Minkowski vacuum, we get

$$\begin{aligned} \left\langle \hat{N}_\Omega \right\rangle &\equiv \langle 0_M|\hat{b}_\Omega^+\hat{b}_\Omega^-|0_M\rangle \\ &= \int_0^\infty d\omega \ |\beta_{\omega\Omega}|^2 \equiv n_\Omega\delta(0) \ . \end{aligned} \tag{10.38}$$

10.5 Bose-Einstein Distribution

We will now show that the occupation number is given by a Bose-Einstein distribution with Rindler-time temperature $T_\tau \equiv a/(2\pi)$. To show this, we need:

$$\begin{aligned} \beta_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}}F(-\omega,\Omega) \\ F(\omega,\Omega) &= F(-\omega,\Omega)\exp\left[\frac{\pi\Omega}{a}\right] \qquad \text{for } \omega>0,\ a>0 \end{aligned} \tag{10.39}$$

To prove (10.39), we start by using the transformation

$$t \equiv -\frac{i\omega}{a}e^{-au} \ . \tag{10.40}$$

With the usual convergence prescription for the oscillatory integral, the integral for F in (10.34) then yields:

$$F(\omega,\Omega) = \frac{1}{2\pi a}\exp\left[i\frac{\Omega}{a}\ln\left(\frac{\omega}{a}\right) + \frac{\pi\Omega}{2a}\right]\Gamma\left(-\frac{i\Omega}{a}\right) \ . \tag{10.41}$$

Changing $\omega \to -\omega$ requires choosing a branch of the complex logarithm. With the principal branch, $\ln(-1) = i\pi$, which gives

$$F(-\omega,\Omega) = e^{-\pi\Omega/a}F(\omega,\Omega) \tag{10.42}$$

and hence

$$F(\omega,\Omega) = F(-\omega,\Omega)\exp\left[\frac{\pi\Omega}{a}\right] \ . \tag{10.43}$$

Now, using the commutator from (10.36) we get:

$$\begin{aligned}
&\int_0^\infty d\omega\left(\alpha_{\omega\Omega}\alpha^*_{\omega\Omega'} - \beta_{\omega\Omega}\beta^*_{\omega\Omega'}\right) = \delta(\Omega-\Omega') \\
\Rightarrow \quad & \delta(\Omega-\Omega') = \int_0^\infty d\omega\frac{\sqrt{\Omega\Omega'}}{\omega}\left[F(\omega,\Omega)F^*(\omega,\Omega') - F(-\omega,\Omega)F^*(-\omega,\Omega')\right] \\
&= \left[\exp\left(\frac{\pi\Omega+\pi\Omega'}{a}\right) - 1\right]\int_0^\infty d\omega\frac{\sqrt{\Omega\Omega'}}{\omega}F(-\omega,\Omega)F^*(-\omega,\Omega') \\
\Rightarrow \quad & \int_0^\infty d\omega\frac{\sqrt{\Omega\Omega'}}{\omega}F(-\omega,\Omega)F^*(-\omega,\Omega') = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1}\delta(\Omega-\Omega') \ .
\end{aligned} \tag{10.44}$$

In the last step, the smooth prefactor multiplying $\delta(\Omega-\Omega')$ is evaluated at $\Omega' = \Omega$.

Now we obtain:

$$\begin{aligned}
\left\langle \hat{N}_\Omega \right\rangle &= \int_0^\infty d\omega\,|\beta_{\omega\Omega}|^2 = \int_0^\infty d\omega\frac{\Omega}{\omega}|F(-\omega,\Omega)|^2 \\
&= \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1}\delta(0) \ .
\end{aligned} \tag{10.45}$$

The delta function here is a continuum-normalisation factor.
Comparing this with the number density from (10.38) where

$$\int_0^\infty d\omega\,|\beta_{\omega\Omega}|^2 \equiv n_\Omega\delta(0) \tag{10.46}$$

we get the occupation number:

$$n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \tag{10.47}$$

10.6 Unruh Temperature

If we now compare our occupation number from (10.47) with a Bose-Einstein distribution, we see that the Minkowski vacuum is thermal with respect to the Rindler Hamiltonian:

$$n(\Omega) = \frac{1}{e^{\Omega/T_\tau} - 1}, \qquad T_\tau = \frac{a}{2\pi} \ . \tag{10.48}$$

The frequency Ω is conjugate to the Rindler coordinate time τ. An observer at fixed ξ instead measures proper time $ds = e^{a\xi} d\tau$, and hence the locally measured energy and temperature are redshifted by the same factor. Since the observer's proper acceleration is $\alpha = a e^{-a\xi}$, the locally measured Unruh temperature is

$$T_U = e^{-a\xi} T_\tau = \frac{\alpha}{2\pi} \ . \tag{10.49}$$

For the particular observer at $\xi = 0$, $\alpha = a$ and therefore $T_U = a/(2\pi)$.

We conclude that a uniformly accelerated observer in the Minkowski vacuum detects a thermal bath with temperature

$$T_U = \frac{\alpha}{2\pi} \tag{10.50}$$

in natural units. This does not mean that inertial observers regard the Minkowski vacuum as containing real particles. Operationally, an accelerated detector can become excited, and the energy required for that excitation is supplied by the external agent that maintains the detector's acceleration.

Is this observable? To illustrate how small this effect is, consider how fast we need to accelerate to boil an egg. To obtain a temperature of $T = 100°C$, such that the observer would see water starting to boil, we need to write the Unruh temperature in SI units:

$$k_B T_U = \frac{\hbar \alpha}{2\pi c} \tag{10.51}$$

From this, we get:

$$T_U = 373\,\mathrm{K} \quad \Rightarrow \quad \alpha \simeq 9 \times 10^{22} \frac{\mathrm{m}}{\mathrm{s}^2} \sim 10^{23} \frac{\mathrm{m}}{\mathrm{s}^2} \ . \tag{10.52}$$

The Unruh effect is an extremely inefficient way to obtain a large detector temperature.[7] However, closely related horizon thermality appears for black holes and de Sitter space-time, as we will see in the next lecture.

[7] and an even more inefficient way to boil eggs

Hawking Radiation

Having seen that an accelerated observer experiences thermal radiation even in the Minkowski vacuum, we will now see that this has deep, seemingly paradoxical consequences for our understanding of black holes. The curvature near the horizon of a black hole is small, but staying at a fixed distance and not falling in requires acceleration, and hence, such an observer near a black hole will experience radiation from the black hole. This leads to an apparent paradox, as we usually say that nothing can escape a black hole. We can understand this phenomenon by expanding the metric near the black hole to leading order in the weak curvature, and then observing that we can approximate it by the Rindler metric in the near-horizon limit.

11.1 Schwarzschild Black Hole

The Schwarzschild geometry describes the vacuum space-time outside a static, spherically symmetric, uncharged mass. Using our $(-+++)$ signature convention, we have the metric:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2MG_N}{r}\right)dt^2 + \left(1-\frac{2MG_N}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \\ &= g_{\mu\nu}dx^\mu dx^\nu , \qquad d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)\, d\phi^2 \ . \end{aligned} \tag{11.1}$$

The time t is the Schwarzschild time, and it represents the proper time recorded by a standard clock at rest at spatial infinity. r is the Schwarzschild radial coordinate.

At

$$r = r_s \equiv 2MG_N \tag{11.2}$$

we have an event horizon. Here $g_{tt} = 0$ and g_{rr} diverges, but this is only a coordinate singularity. There is a true curvature singularity at $r = 0$. For example, the Kretschmann scalar is

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = \frac{48G_N^2M^2}{r^6}, \tag{11.3}$$

which is finite at $r = r_s$ but diverges at $r = 0$. An observer in free fall through the horizon sees no local geometrical singularity at $r = r_s$ provided the black hole is sufficiently large that tidal forces there are small. However, to a distant Schwarzschild observer, an infalling object approaches the horizon only asymptotically in the coordinate time t and becomes increasingly redshifted.

For radial light following a null geodesic, $ds^2 = 0$, we have in Schwarzschild coordinates

$$\left|\frac{dr}{dt}\right| = 1 - \frac{2MG_N}{r}, \qquad r > 2MG_N \ . \tag{11.4}$$

The velocity of light is approaching zero as $r \to 2MG_N$ from the point of view of a distant observer. Also, since $dt = \frac{ds}{\sqrt{g_{00}}}$ and dS is invariant, there is an infinite redshift of frequency $\nu = \frac{2\pi}{dt}$ as $g_{00} \to 0$

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}\left(x_2\right)}{g_{00}\left(x_1\right)}} \to 0 \qquad \text{for } g_{00}\left(x_2\right) \to 0 \tag{11.5}$$

11.2 Near Horizon Limit

Define ρ as the proper radial distance to the horizon along a hypersurface of constant Schwarzschild time:

$$\begin{aligned}\rho &= \int_{2MG_N}^{r} \sqrt{g_{rr}\left(r'\right)}dr' = \int_{2MG_N}^{r} \left(1 - \frac{2MG_N}{r'}\right)^{-\frac{1}{2}} dr' \\ &= \sqrt{r\left(r - 2MG_N\right)} + 2MG_N \sinh^{-1}\left(\sqrt{\frac{r}{2MG_N} - 1}\right) \ .\end{aligned} \tag{11.6}$$

Near the horizon:

$$\begin{aligned}&\rho \approx 2\sqrt{2MG_N\left(r - 2MG_N\right)} \\ \Rightarrow \quad & r - 2MG_N \approx \frac{\rho^2}{8MG_N} \ .\end{aligned} \tag{11.7}$$

Therefore

$$1 - \frac{2MG_N}{r} \approx \frac{\rho^2}{16M^2G_N^2} \tag{11.8}$$

and the near-horizon metric becomes

$$ds^2 \cong -\rho^2 \left(\frac{dt}{4MG_N}\right)^2 + d\rho^2 + r_s^2 d\Omega^2 \ . \tag{11.9}$$

It is useful to introduce the surface gravity

$$\kappa \equiv \frac{1}{4MG_N} = \frac{1}{2r_s} \ . \tag{11.10}$$

Then the two-dimensional (t, ρ) part of the metric is

$$ds^2_{(2)} \cong -\kappa^2 \rho^2 dt^2 + d\rho^2 \ , \tag{11.11}$$

which is precisely the Rindler form.

Using the parameter ξ defined by

$$\rho = \frac{1}{\kappa} e^{\kappa \xi}, \tag{11.12}$$

we obtain

$$ds^2_{(2)} \cong e^{2\kappa\xi} \left(-dt^2 + d\xi^2\right) \ . \tag{11.13}$$

A static observer at fixed ρ, or equivalently fixed ξ, has in the near-horizon region the proper acceleration

$$\alpha \simeq \frac{1}{\rho} = \kappa e^{-\kappa\xi} \ . \tag{11.14}$$

The exact proper acceleration of a static Schwarzschild observer at radius r is

$$\alpha(r) = \frac{G_N M}{r^2 \sqrt{1 - \frac{2G_N M}{r}}}, \tag{11.15}$$

which diverges as the horizon is approached. The finite black-hole surface gravity, $\kappa = 1/(4G_N M)$, is therefore not the proper acceleration of an observer sitting at the horizon, but it can be thought of as the redshifted acceleration measured at infinity:

$$\sqrt{1 - \frac{r_s}{r}}\, \alpha(r) = \kappa \ . \tag{11.16}$$

Thus, surface gravity is a non-local concept. It is the finite force per unit mass that an observer at infinity must exert on a "rope" to hold a test mass stationary right at the black hole's edge.

11.3 Hawking Temperature

The local Rindler argument gives a static observer close to the horizon a local Unruh temperature

$$T_{\rm loc} = \frac{\alpha}{2\pi} \simeq \frac{1}{2\pi\rho} \ . \tag{11.17}$$

This local temperature diverges as $\rho \to 0$. However, radiation climbing out of the gravitational potential is redshifted. The temperature measured by an observer at infinity is therefore

$$T_H = \sqrt{1 - \frac{r_s}{r}}\, T_{\rm loc} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M G_N} \ . \tag{11.18}$$

This is the Hawking temperature.

For an ideal bosonic mode, the thermal occupation factor is

$$n_E = \frac{1}{e^{E/T_H} - 1} \ . \tag{11.19}$$

In four-dimensional Schwarzschild space-time, the spectrum measured at infinity is not an exact blackbody spectrum because the modes scatter from the curvature outside the horizon. These frequency- and angular-momentum-dependent transmission probabilities are called greybody factors. The Hawking temperature nevertheless remains $T_H = \kappa/(2\pi)$.

As with the Unruh temperature, the Hawking temperature is a very small effect for astrophysical black holes. As an example, for a solar-mass black hole:

$$\begin{aligned} M = M_\odot \approx 2 \cdot 10^{30}\text{kg}, \qquad r_s \simeq 3\,\text{km} \\ \Rightarrow \quad T_H \approx 6 \cdot 10^{-8}\text{K} \ . \end{aligned} \tag{11.20}$$

It is a tiny effect, but an isolated black hole in an environment colder than its Hawking temperature will lose mass through Hawking radiation. Astrophysical black holes immersed in the present cosmic microwave background are colder than their surroundings and therefore do not currently undergo net evaporation.

11.4 De Sitter Space

We can also, in a very similar way, associate a temperature with the horizon of a static observer in de Sitter space. By a coordinate transformation, it can be shown that the de Sitter metric can be written in so-called static coordinates, where the metric resembles the metric of a black hole "turned inside out". The de Sitter metric takes the form

$$ds^2 = -\left(1 - \frac{r^2}{r_s^2}\right) dt^2 + \left(1 - \frac{r^2}{r_s^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{11.21}$$

where

$$r_s = \frac{1}{H} \tag{11.22}$$

is the de Sitter radius. The cosmological horizon is at $r = r_s$. Its surface gravity is

$$\kappa_{dS} = H \ , \tag{11.23}$$

and therefore the Gibbons–Hawking temperature is

$$T_{dS} = \frac{H}{2\pi} \ . \tag{11.24}$$

This is closely related to the characteristic amplitude of a light scalar fluctuation in de Sitter space. In particular, the dimensionless power satisfies

$$P_{\delta\phi}^{1/2} \simeq \frac{H}{2\pi}, \tag{11.25}$$

whereas the individual Fourier mode behaves as $|\delta\phi_k| \simeq H/\sqrt{2k^3}$.

11.5 Kruskal versus Boulware Vacuum

Let us consider again the radial part of the Schwarzschild solution:

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 \ . \tag{11.26}$$

We will now go to the tortoise coordinate $r^*(r)$ defined by:

$$\begin{aligned} dr^* &= \frac{dr}{1 - \frac{r_s}{r}} \\ \Rightarrow \quad r^*(r) &= r - r_s + r_s \ln\left(\frac{r}{r_s} - 1\right) , \end{aligned} \tag{11.27}$$

where the additive constant $-r_s$ is conventional and physically irrelevant.

With the tortoise coordinate, the metric becomes:

$$ds^2 = \left(1 - \frac{r_s}{r(r^*)}\right)\left(-dt^2 + dr^{*2}\right) \ . \tag{11.28}$$

Now, defining the light-cone coordinates:

$$\begin{aligned} &\bar{u} \equiv t - r^*, \qquad \bar{v} \equiv t + r^* \\ \Rightarrow \quad &ds^2 = -\left(1 - \frac{r_s}{r}\right) d\bar{u}\, d\bar{v} \ . \end{aligned} \tag{11.29}$$

Equivalently, we can define Kruskal light-cone coordinates:

$$u = -2r_s e^{-\frac{\bar{u}}{2r_s}}, \qquad v = 2r_s e^{\frac{\bar{v}}{2r_s}} \ . \tag{11.30}$$

Using the definition of r^* above, the radial metric becomes

$$ds^2 = -\frac{r_s}{r} e^{1-r/r_s} du\, dv \ , \tag{11.31}$$

which is manifestly finite and nonzero at $r = r_s$. Thus the Schwarzschild-coordinate singularity at the horizon has disappeared.

For a massless scalar in an effectively $1+1$ dimensional near-horizon treatment, the conformal factor does not affect the classical mode equation. In the full four-dimensional problem, each angular momentum mode also experiences an effective scattering potential; this is the origin of the greybody factors mentioned above.

The relation between $(\bar{u}, \bar{v})$ and (u, v) contains the same exponential structure as the Minkowski-to-Rindler transformation, with

$$\begin{aligned} \kappa &= \frac{1}{2r_s} = \frac{1}{4G_N M} \\ \Rightarrow \quad T_H &= \frac{\kappa}{2\pi} = \frac{1}{8\pi G_N M} \ . \end{aligned} \tag{11.32}$$

11.6 Vacuum Considerations

Let us compare the Unruh effect and the Hawking radiation. In both cases, we have two vacuums. For the Rindler transformation we had one for the inertial observer $|0_M\rangle$ (Minkowski vacuum) and one for the accelerated observer $|0_R\rangle$ (Rindler vacuum), likewise for the Schwarzschild we have a vacuum for the free falling observer $|0_K\rangle$ (Kruskal vacuum) and one for the observer at constant r $|0_B\rangle$ (Boulware vacuum) respectively.

Furthermore, for the Rindler transformation we had the acceleration a, while in the Schwarzschild case it depends on the mass of the black hole: $\frac{1}{4MG_N}$.

In the Minkowski/Kruskal frame, the energy density in the $|0_R\rangle/|0_B\rangle$ state diverges near the horizon. The Boulware vacuum is defined using positive frequency with respect to Schwarzschild time. It looks empty to static observers at infinity, but its renormalised stress-energy tensor diverges at the event horizon. It is therefore not an appropriate state for a black hole formed by gravitational collapse.

For an eternal Schwarzschild black hole, the Kruskal vacuum (also called the Hartle–Hawking state) is regular on both the future and past horizons and describes a black hole in thermal equilibrium with a bath of incoming and outgoing radiation at the Hawking temperature. Positive-frequency modes that are regular in Kruskal coordinates are naturally associated with this horizon-regular construction.

The eternal Schwarzschild black hole is an idealisation, and for a black hole formed by collapse, the physically relevant state is instead the Unruh vacuum. It contains no incoming thermal radiation from past null infinity, is regular on the future event horizon, and produces an outgoing Hawking flux at future null infinity.

We therefore have the analogy

Rindler	**Schwarzschild**
inertial Minkowski vacuum $\|0_M\rangle$	horizon-regular state
Rindler vacuum $\|0_R\rangle$	Boulware vacuum $\|0_B\rangle$
local proper acceleration α	$\alpha(r) = \dfrac{G_N M}{r^2\sqrt{1-r_s/r}}$
Rindler scale a	surface gravity $\kappa = \dfrac{1}{4G_N M}$

The surface gravity κ is finite, whereas the proper acceleration of a static observer diverges at the horizon. The redshift of the divergent local Unruh temperature converts it into the finite Hawking temperature $T_H = \kappa/(2\pi)$ seen at infinity.

Newtonian Cheat

Using Newtonian gravity, we can evaluate the gravitational acceleration at the Schwarzschild radius $r_s = 2G_N M$:

$$g = \frac{G_N M}{r_s^2} = \frac{1}{4G_N M} \ . \tag{11.33}$$

Numerically, this equals the Schwarzschild surface gravity

$$g = \kappa = \frac{1}{4G_N M}. \tag{11.34}$$

Inserting this surface-gravity scale into the Unruh formula happens to give

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi G_N M} \ . \tag{11.35}$$

This is a quick way to remember the result, though it is not a derivation of Hawking radiation.

11.7 Black Hole Thermodynamics

For a static Schwarzschild black hole, the horizon area and Hawking temperature are[8]

$$A = 4\pi r_s^2 = 16\pi M^2, \qquad T_H = \frac{1}{8\pi M} \ . \tag{11.36}$$

To get an idea of how this works thermodynamically, we can use the Stefan–Boltzmann law as a rough estimate for the flux of energy radiated from the black hole:

$$L \approx \gamma \sigma T_H^4 A \ . \tag{11.37}$$

Where

$$\sigma = \frac{\pi^2}{60} \tag{11.38}$$

is the Stefan–Boltzmann constant for one bosonic degree of freedom in natural units. The factor γ schematically accounts for the species that can be emitted. In an actual black-hole calculation, the greybody factors modify both the spectrum and the numerical coefficient, so the following estimate should be viewed as illustrative rather than exact.

Using the ideal blackbody approximation,

$$L \approx \frac{\gamma}{15360\pi M^2} \tag{11.39}$$

and therefore

$$\begin{aligned} \frac{dM}{dt} &= -L = -\frac{\gamma}{15360\pi M^2} \\ \Rightarrow \quad M(t) &= M_0 \left(1 - \frac{t}{t_L}\right)^{\frac{1}{3}}, \qquad t_L \equiv 5120\pi \frac{M_0^3}{\gamma} \ . \end{aligned} \tag{11.40}$$

Where M_0 is the initial mass of the black hole and t_L is the approximate lifetime in this simplified treatment. Since $T_H \propto 1/M$, the black hole becomes hotter as it loses mass. Eventually the semiclassical approximation ceases to be reliable when the mass approaches the quantum-gravity scale.

[8] In the remainder of this section we use natural Planck units $G_N = \hbar = c = k_B = 1$ unless factors of G_N are explicitly restored.

Laws of Black Hole Thermodynamics

Entropy is statistically related to the number of accessible microscopic states,

$$S = \log(N) \tag{11.41}$$

in units with $k_B = 1$.

In the early 1970s, Bekenstein proposed that a black hole should carry an entropy proportional to its horizon area. Hawking's temperature fixes the proportionality constant through the first law

$$dE = dM = T_H dS_{BH} \ . \tag{11.42}$$

If

$$S_{BH} = \alpha A = 16\pi\alpha M^2 \ , \tag{11.43}$$

then

$$\begin{aligned} 1 = T_H \frac{dS_{BH}}{dM} &= \frac{1}{8\pi M}(32\pi\alpha M) = 4\alpha \\ \Rightarrow \quad \alpha &= \frac{1}{4} \ . \end{aligned} \tag{11.44}$$

This gives the Bekenstein–Hawking entropy:

$$S_{BH} = \frac{A}{4} = 4\pi M^2 \ . \tag{11.45}$$

Restoring constants,

$$S_{BH} = \frac{k_B c^3 A}{4 G_N \hbar} \ . \tag{11.46}$$

Black hole equation of state:

$$E(T) = M = \frac{1}{8\pi T} \ . \tag{11.47}$$

The heat capacity of an asymptotically flat Schwarzschild black hole is negative:

$$C_{BH} = \frac{\partial E}{\partial T} = -\frac{1}{8\pi T^2} < 0 \ . \tag{11.48}$$

This means that a black hole becomes colder when it absorbs energy and hotter when it loses energy. Consequently, an asymptotically flat Schwarzschild black hole is not thermodynamically stable in the usual canonical ensemble with an infinite heat bath.

Generalized Second Law of Thermodynamics (GSL)

The generalised entropy is the sum of the black-hole entropy and the entropy outside the horizon,

$$S_{\text{gen}} = S_{\text{outside}} + S_{BH} \ . \tag{11.49}$$

The generalised second law states

$$\delta S_{\text{gen}} \geq 0 \ . \tag{11.50}$$

Classically, under appropriate energy conditions, Hawking's area theorem states that the area of a black-hole event horizon cannot decrease. Hawking evaporation is a quantum process and therefore lies outside the assumptions of the classical area theorem, and the decrease of S_{BH} is compensated by entropy carried by the Hawking radiation in accordance with the generalised second law.

Information Paradox

Consider the collapse of ordinary matter into a black hole. In a local quantum field theory, the entropy scales like the volume, and therefore initially, the entropy/number of degrees of freedom scales like volume. After the collapse, the entropy/number of degrees of freedom of the black hole scales like area. This leads to information being lost, and GSL being violated for large enough volumes. In rough terms, the two possible means of resolving this are historically: 1) Give up unitarity (Hawking). 2) Give up locality ('t Hooft, Susskind).

More precisely, the paradox comes from combining gravitational collapse with Hawking's semiclassical calculation. Suppose an initially pure quantum state collapses to form a black hole. Hawking's calculation predicts outgoing radiation that is approximately thermal and, in the semiclassical approximation, appears to be entangled with partner modes behind the horizon. If the black hole evaporates completely and the radiation remains exactly thermal, an initially pure state would apparently evolve into a mixed state, violating unitary quantum evolution.

The tension therefore involves several principles: unitarity of quantum mechanics, the semiclassical description of a smooth horizon, and ordinary local quantum field theory. Modern approaches to quantum gravity generally seek to preserve unitarity, implying that the semiclassical description must receive subtle corrections. Ideas include holography, black-hole complementarity, nonlocal quantum-gravitational effects, and more recent calculations based on quantum extremal surfaces and islands.

Holography

The fact that the Bekenstein–Hawking entropy scales with horizon area rather than spatial volume was one of the main motivations for the holographic principle: a gravitational system may contain no more independent information than can be encoded on an appropriate boundary surface.

A simple inequality of the form

$$S \lesssim \frac{A}{4} \tag{11.51}$$

is suggestive in Planck units but is not a universally valid entropy bound for arbitrary spatial regions. A more precise gravitational formulation is provided by covariant entropy bounds defined on suitable light-sheets. Gauge/gravity dualities such as AdS/CFT provide concrete realisations of the broader holographic idea.

Short Interlude on General Relativity

Remember that in special relativity (SR), products of four-vectors are defined using the Minkowski metric $\eta_{\mu\nu}$. The Minkowski metric describes a flat space-time geometry. In general relativity (GR), the Minkowski metric is replaced by a general space-time metric

$$\eta_{\mu\nu} \to g_{\mu\nu}(x) \tag{A.1}$$

which need not be related to $\eta_{\mu\nu}$ by a global coordinate transformation. The metric becomes a dynamical field, and its dynamics are determined by Einstein's equation.

The vacuum Einstein equation can also be obtained as an Euler–Lagrange equation by varying the Einstein–Hilbert action with respect to the metric. To describe gravity as a consequence of the geometry of space-time, we can therefore apply the principle of stationary action to the Einstein–Hilbert action. In this section we will briefly go through the essentials of GR necessary to get a rudimentary understanding of the Einstein–Hilbert action and Einstein's equation.

A.1 Covariant and Contravariant Tensors

The laws of physics can be written as tensor equations whose form is valid in arbitrary coordinate systems. This property is usually called general covariance. One should distinguish covariance under a change of coordinates from the statement that individual tensor components are invariant: the components generally change, while the geometrical tensor does not depend on the coordinates used to describe it.

Let us consider a coordinate transformation

$$x \to x' \quad \Leftrightarrow \quad dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \ . \tag{A.2}$$

We then define scalars, vectors and tensors according to their transformation properties:

$$\begin{aligned}
\text{Scalar:} \qquad & \phi'(x') = \phi(x) && (\phi \to \phi') \\
\text{Contravariant vector:} \qquad & U'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} U^\nu(x) && (U^\mu \to U'^\mu) \\
\text{Covariant vector:} \qquad & A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) && (A_\mu \to A'_\mu) \ .
\end{aligned} \tag{A.3}$$

The coordinate transformation of a general tensor with p contravariant and q covariant indices is

$$T'^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\sigma_p}}\frac{\partial x^{\rho_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\rho_q}}{\partial x'^{\nu_q}}T^{\sigma_1\dots\sigma_p}_{\rho_1\dots\rho_q}(x) \ . \tag{A.4}$$

A.2 The Metric

The equivalence principle implies that at any space-time event P we can choose locally freely falling coordinates y^α in which

$$g^{(y)}_{\alpha\beta}(P) = \eta_{\alpha\beta}, \qquad \partial_\gamma g^{(y)}_{\alpha\beta}(P) = 0 \ . \tag{A.5}$$

Thus gravity can be transformed away locally at a single event, although tidal effects associated with curvature generally remain.

For a timelike interval in these locally inertial coordinates,

$$d\tau^2 = -\eta_{\alpha\beta}dy^\alpha dy^\beta \ . \tag{A.6}$$

If x^μ are arbitrary coordinates describing the same event, then at P

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}dx^\mu dx^\nu = -g_{\mu\nu}(P)dx^\mu dx^\nu \tag{A.7}$$

with

$$g_{\mu\nu}(P) = \eta_{\alpha\beta}\left.\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\right|_P \ . \tag{A.8}$$

This relation is a local statement at P. If one could write a metric as the pullback of $\eta_{\alpha\beta}$ by one coordinate transformation throughout a finite region, that region would be flat. A genuinely curved space-time cannot in general be obtained by merely making a global coordinate transformation of Minkowski space.

The metric is a covariant rank-two tensor and therefore transforms as

$$\boxed{g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) \ .} \tag{A.9}$$

The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ lower and raise indices. For example,

$$T_{\mu\nu} = g_{\mu\sigma}g_{\nu\rho}T^{\sigma\rho} \ . \tag{A.10}$$

The ordinary derivative of a tensor is not in general a tensor under arbitrary coordinate transformations. We therefore introduce the covariant derivative. For a covector V_ν,

$$D_\mu V_\nu = \frac{\partial V_\nu}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} V_\sigma \ . \tag{A.11}$$

For example, the gradient of a scalar,

$$V_\mu = \frac{\partial \phi}{\partial x^\mu}, \tag{A.12}$$

is a covector.

For the torsion-free, metric-compatible Levi–Civita connection, the Christoffel symbols are

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\sigma} \left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right] \ . \tag{A.13}$$

The Christoffel symbols are not themselves components of a tensor. Curvature is instead described by the Riemann tensor. We use here the convention

$$\boxed{\mathcal{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \ .} \tag{A.14}$$

With this convention and the $(-+++)$ metric signature, Einstein's equation below takes the standard form $G_{\mu\nu} = +8\pi G_N T_{\mu\nu}$. The components of the Riemann tensor are coordinate-dependent, but the statement that the tensor vanishes is coordinate-independent; vanishing Riemann curvature implies local flatness.

The Ricci tensor is obtained by contraction:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\lambda{}_{\mu\lambda\nu} \ . \tag{A.15}$$

Equivalently, if

$$\mathcal{R}_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} \mathcal{R}^\sigma{}_{\mu\nu\kappa}, \tag{A.16}$$

then

$$\mathcal{R}_{\mu\kappa} = g^{\lambda\nu} \mathcal{R}_{\lambda\mu\nu\kappa} = \mathcal{R}^\lambda{}_{\mu\lambda\kappa} \ . \tag{A.17}$$

The Ricci scalar is

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} \ . \tag{A.18}$$

The Einstein tensor is

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} \ . \tag{A.19}$$

The contracted Bianchi identity implies

$$\boxed{D^{\mu}G_{\mu\nu} = 0 \ .} \tag{A.20}$$

The field equation that relates geometry to matter is

$$\boxed{G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ .} \tag{A.21}$$

This is Einstein's equation, and $T_{\mu\nu}$ is the energy–momentum tensor. Consistency with the Bianchi identity implies

$$D^{\mu}T_{\mu\nu} = 0 \ . \tag{A.22}$$

$G_{\mu\nu}$ describes the curvature of space-time produced by the matter and energy described by $T_{\mu\nu}$.

A.3 Einstein–Hilbert Action

Einstein's equation can be derived from the Einstein–Hilbert action. Let

$$g \equiv \det(g_{\mu\nu}) \ . \tag{A.23}$$

For a Lorentzian metric with signature $(-+++)$, $g < 0$, so the invariant volume element is

$$\boxed{d^4x\,\sqrt{-g} \ .} \tag{A.24}$$

The simplest local scalar containing two derivatives of the metric is the Ricci scalar $\mathcal{R}$. We therefore define

$$S_H = \int d^4x\sqrt{-g}\,\mathcal{R} \ . \tag{A.25}$$

Varying this action with respect to the inverse metric gives, after discarding the total-derivative boundary term,

$$\boxed{\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = G_{\mu\nu} \ .} \tag{A.26}$$

Hence in vacuum,

$$G_{\mu\nu} = 0 \ . \tag{A.27}$$

For a space-time with a boundary, a well-posed metric variational problem also requires the appropriate Gibbons–Hawking–York boundary term. For the present discussion, we suppress this boundary contribution.

Now add matter to the action:

$$\boxed{S = \frac{1}{16\pi G_N} S_H + S_M \ .} \tag{A.28}$$

The energy–momentum tensor is defined by

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \ .} \tag{A.29}$$

Equivalently,

$$\delta S_M = -\frac{1}{2} \int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu} \ . \tag{A.30}$$

The metric variation of the full action is therefore

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G_N} G_{\mu\nu} - \frac{1}{2} T_{\mu\nu} = 0 \tag{A.31}$$

which gives

$$\boxed{G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ .} \tag{A.32}$$

The matter energy–momentum tensor is covariantly conserved,

$$D^\mu T_{\mu\nu} = 0 \ . \tag{A.33}$$

For a diffeomorphism-invariant matter action, this conservation law follows from the matter equations of motion together with diffeomorphism invariance; it is also required by Einstein's equation and the contracted Bianchi identity.

Finally, we can also include a cosmological constant. The gravitational action becomes

$$\boxed{S_{\text{grav}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, (\mathcal{R} - 2\Lambda) \ .} \tag{A.34}$$

Including matter, the Einstein equation is

$$\boxed{\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}\ .} \tag{A.35}$$

In vacuum, this reduces to

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{A.36}$$

or

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}\ . \tag{A.37}$$

The cosmological constant can equivalently be moved to the matter side and interpreted as a vacuum stress-energy tensor,

$$\boxed{T^{(\Lambda)}_{\mu\nu} = -\rho_\Lambda g_{\mu\nu}, \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G_N}, \qquad p_\Lambda = -\rho_\Lambda\ .} \tag{A.38}$$